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Grounding Generalizations

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Abstract
Some propositions are true, and it is true that some propositions are true. Each of these facts looks like an impeccable ground of the other. But they cannot both ground each other, since grounding is asymmetric. This paper explores two new diagnoses of this much discussed puzzle. The tools of higher-order logic are used to show how both diagnoses can be fleshed out into strong and consistent theories of grounding. These theories of grounding in turn demand new theories of the granularity of propositions, properties, and relations. Even those who are uninterested in grounding should take seriously these pictures of reality's logical structure, which are in many ways reminiscent of Russell's and Wittgenstein's logical atomism.

Keywords Grounding · Paradox · Higher-order logic · Fineness of grain

1 The Puzzle

Say that p *grounds* q just in case q is true at least partially in virtue of p. Recent years have seen an explosion of work on this and closely related notions of grounding, often understood as a kind of metaphysical explanation.[1]

Two central tenets of this literature are (i) that nothing grounds itself and (ii) that true generalizations are grounded in their true instances. Together (i) and (ii) present a puzzle. Consider the claim that something is true. It is true. And it seems to be a generalization of which it is itself an instance. So (ii) seems to require it to ground itself, which (i) prohibits.[2]

[1] I am operating here with Fine's ([15]) notion of (factive mediate) strict partial ground.
[2] Fine [14] initiated discussion of this and related puzzles; for a survey see [28].

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Following Krämer [25], we can regiment this argument in a formal language with a binary connective $\prec$ expressing the operative notion of grounding, variables $p, q, \ldots$ of the same syntactic category as sentences, and corresponding quantifier prefixes $\exists p, \forall p, \exists q, \forall q, \ldots$ binding these variables. In this language the idea that existential generalizations are grounded in their true instances can be regimented schematically as follows:

$$\forall p(\varphi \rightarrow (\varphi \prec \exists p\varphi)). \tag{1}$$

Letting φ simply be the formula p, we obtain the following instance:

$$\forall p(p \rightarrow (p \prec \exists pp)). \tag{2}$$

Instantiating the variable p with the sentence $\exists pp$ then yields:

$$\exists pp \rightarrow (\exists pp \prec \exists pp). \tag{3}$$

The antecedent is true. By modus ponens, we have:

$$\exists pp \prec \exists pp. \tag{4}$$

But now consider the claim that nothing grounds itself (i.e., that grounding is irreflexive):

$$\forall p\neg(p \prec p). \tag{5}$$

Instantiating p with $\exists pp$ yields the negation of (4). We have a contradiction.

Fritz [17] suggests an intriguing response to this puzzle. He argues that (1) is not the best way to formalize the idea that true existential generalizations are grounded in their true instances. He proposes a different way of regimenting that idea, which *is* compatible with the irreflexivity of grounding. This different regimentation requires theorizing in a higher-order language in which quantification is not formalized using variable-binding sentential operators like $\exists p$, $\forall q$, etc. Let me explain.

Krämer's argument is formulated in a language where quantifier prefixes like $\exists p$ do double-duty: they both bind variables and express a kind of generality. In higher-order logic we can separate these two roles. The idea goes back to Frege, and it is standard both in natural-language semantics and in many applications of higher-order languages in contemporary metaphysics. Rather than having infinitely many quantifier prefixes $\exists p, \exists q, \ldots$, we simply have a single existential quantifier $\exists$ that combines with a monadic sentential operator like $\neg$ (i.e., something that itself combines with a formula to form a formula) to form a formula. For example, $\exists\neg$ is a sentence,[3] which we might pronounce "negation is instantiated", although this gloss is not meant to suggest that there is any general recipe for translating sentences of our higher-order language into English. Variable-binding is done with λ-abstraction. If φ is a formula, $(\lambda p.\varphi)$ is a monadic sentential operator in which all free occurrences of the variable p in φ are bound by the prefix λp. So instead of having sentences like $\exists p\varphi$, we now have sentences like $\exists(\lambda p.\varphi)$.

So far we have only considered variables of the same syntactic category as sentences. But it is standard in higher-order logic to allow bindable variables of other

[3] We allow context to disambiguate use and mention where no confusion is likely to arise.

syntactic categories. These categories are called *types*. We will work in a simply relationally typed language. e is a type (the type of singular terms); every sequence of types $\langle \tau_1, \ldots, \tau_n \rangle$ is a type; nothing else is a type. An expression of type $\langle \tau_1, \ldots, \tau_n \rangle$ can be thought of as an n-place predicate, which combines with n arguments respectively of types $\tau_1, \ldots, \tau_n$ to form a formula. Formulas themselves have type $\langle\rangle$: they don't need to be combined with any arguments in order to form a formula. Monadic sentential operators like $\neg$ and $(\lambda p.p)$ are of type $\langle\langle\rangle\rangle$: they combine with a formula to form a formula. The existential quantifier that combines with such an operator to form a formula is therefore of type $\langle\langle\langle\rangle\rangle\rangle$. More generally, for every type τ, we have existential and universal quantifiers $\exists_\tau$ and $\forall_\tau$ of type $\langle\langle\tau\rangle\rangle$. And for any type $\langle \tau_1, \ldots, \tau_n \rangle$ (where $n > 0$), we can form terms of that type by λ-abstraction: if φ is a formula, and $x_1, \ldots, x_n$ are pairwise-distinct variables respectively of types $\tau_1, \ldots, \tau_n$, then $(\lambda x_1 \ldots x_n.\varphi)$ is a term of type $\langle \tau_1, \ldots, \tau_n \rangle$ in which all free occurrences in φ of any of $x_1, \ldots, x_n$ are bound. The type of a variable is sometimes indicated by a superscript on its first occurrence in a formula. For readability, we will continue to deploy the notation of quantifier prefixes, but only as a convenient abbreviation – for example, $\forall x^\tau \varphi$ is now shorthand for $\forall_\tau(\lambda x.\varphi)$.

In this language, the dictum that true existential generalizations are grounded in their true instances can be regimented schematically as follows:

$$\forall F^{\langle\tau\rangle} \forall x^\tau (Fx \rightarrow (Fx \prec \exists_\tau F)). \tag{6}$$

This schema has one instance for every type τ. The instance involving quantification into sentence position is:

$$\forall F^{\langle\langle\rangle\rangle} \forall p^{\langle\rangle} (Fp \rightarrow (Fp \prec \exists_{\langle\rangle} F)). \tag{7}$$

Unlike (1), this generalization *does* seem to be consistent with the irreflexivity of grounding. To illustrate, note that $\exists pp$ now abbreviates the sentence $\exists_{\langle\rangle}(\lambda p.p)$. Instantiating F and p in (7) with $(\lambda p.p)$ and $\exists(\lambda p.p)$ and performing modus ponens then yields not (4) but rather:

$$(\lambda p.p)\exists(\lambda p.p) \prec \exists(\lambda p.p) \tag{8}$$

And this is not inconsistent with the irreflexivity of grounding, at least not without further assumptions.

Now there are two natural such assumptions either of which would render (8) inconsistent with the irreflexivity of grounding. The first is the orthodox principle of *β-conversion*, according to which formulas of the form $(\lambda v_1 \ldots v_n.\varphi)a_1 \ldots a_n$ and $\varphi[a_i/v_i]$ are everywhere intersubstitutable.[4] This principle would allow us to replace $(\lambda p.p)\exists(\lambda p.p)$ with $\exists(\lambda p.p)$ in (8), generating a case of self-grounding.

The second such assumption is:

$$\forall p(p \rightarrow p \prec (\lambda p.p)p). \tag{9}$$

[4]More carefully, they are intersubstitutable provided no variable free in any a_i becomes bound when substituted for any free occurrence of x_i in φ. ($\varphi[a_i/v_i]$ denotes the result of simultaneously performing all such substitutions.)

Instantiating p with $\exists(\lambda p.p)$ and performing modus ponens would yield:

$$\exists(\lambda p.p) \prec (\lambda p.p)\exists(\lambda p.p). \tag{10}$$

Given (8), this would be a counterexample to the asymmetry of grounding, and hence assuming the transitivity of grounding (which I will not question here), a counterexample to the irreflexivity of grounding.

There are two reasons why (9) might be appealing to grounding theorists. One is that it is a natural regimentation of the intuitive idea that any truth grounds the fact that it is true (where $(\lambda p.p)$ is taken to regiment the sentential operator "it is true that . . ."). Another is the principle of *β-grounding*, endorsed by [15], according to which truths expressed by predications of λ-terms are grounded in truths expressed by the corresponding β-reduced formulas (schematically: $\varphi[a_i/x_i] \rightarrow \varphi[a_i/x_i] \prec (\lambda x_1 \ldots x_n.\varphi)a_1 \ldots a_n)$).

While there is no consensus about which of β-conversion and (9) to accept, grounding theorists typically accept one or the other.[5] This sociological fact notwithstanding, Fritz argues that there could be a principled basis for rejecting both β-conversion and (9), as his response to Krämer's puzzle requires. That basis is a certain picture of propositional granularity, according to which "the application of properties specified using λ-terms obliterates [propositions'] structural content, but does not introduce any structural content itself".

This is an intriguing proposal. Notice that it involves denying that propositions are structured in the manner of the sentences that express them. For while a sentence $\ulcorner(\lambda x_1 \ldots x_n.\varphi)a_1 \ldots a_n)\urcorner$ is more syntactically complex than the sentence $\varphi[a_i/x_i]$ that it immediately β-reduces to, the proposition it expresses will have no structure at all, and hence will have no more and perhaps even less structure than the propositions expressed by the sentence it immediately β-reduces to.[6] In light of this fact, it is unclear whether the proposal can be fleshed out in such a way that propositions are still fine-grained enough to support a robust theory of grounding. As Fritz writes:

> [I]t is not clear whether developing the present picture in natural ways will actually lead to a consistent theory of propositional granularity and grounding at all. The suggestions made here are merely supposed to illustrate how the surprising commitments concerning grounding we have ended up with may in principle arise as natural consequences of interesting views about the individuation of propositions.

Here is the plan for the paper. In Section 2 I will show that a version of Fritz's suggestion can indeed be developed into a consistent theory. Section 3 refines the theory to make it more amenable to familiar ways of thinking about the structure of logical space. In many ways the resulting theory is quite attractive. Section 4 argues that it offers a more promising response to Krämer's puzzle than available solutions

[5] They cannot accept both because the two principles are jointly inconsistent with the irreflexivity of grounding, since (9) is β-equivalent to the claim that every truth grounds itself.

[6] Note that the most straightforward formulation of the idea that propositions are structured in the image of the sentences expressing them is inconsistent on its own, given weak logical assumptions having nothing to do with grounding; see [22] and Section 6.

in the literature do, once these competing proposals' implications for propositional granularity are made explicit. But there are costs. The irreflexivity of grounding, together with orthodox principles about the ground-theoretic behavior of quantifiers and Boolean connectives, forces proponents of the theory either to draw an invidious distinction between binary and unary logical connectives, or to deny that there are any such relations as conjunction and disjunction. These limitative results are discussed in Section 5. Section 6 develops another theory of grounding, based on a different picture of propositional granularity, which avoids these limitations by responding to Krämer's puzzle in the opposite way. It holds that not all generalizations have 'quantificational structure', and that only when they do must true generalizations be grounded in their true instances. Formal models of these theories are given in appendices.

The theories of reality's granularity developed here are of interest independently of the considerations of grounding that occasioned them. Talk of propositions, or facts, having other propositions(/facts) as conjuncts, disjuncts, or instances is ubiquitous in philosophy. But as we will see, our naïve way of thinking about this kind of logical structure is inconsistent. The theories developed here display the tradeoffs inherent in trying to hold on to as much of that naïve conception as possible. Section 7 amplifies these morals, and shows how ideas arising separately as components of competing diagnoses of grounding puzzles can be integrated into an attractive form of logical atomism.

2 A Theory of Ground and Grain

I'll begin by describing a picture of propositional granularity that vindicates a version of Fritz's proposal. For ease of exposition I will speak as if reality consists of a single domain of entities which can be classified into different types, such as individuals (type e), propositions (type $\langle\rangle$), monadic properties of propositions (type $\langle\langle\rangle\rangle$), binary relations between individuals (type $\langle e, e\rangle$), etc. But this is merely an expository convenience. Officially, type distinctions are syntactic distinctions among expressions of our higher-order language. Generalizations about entities of various types are shorthand for quantification over elements of the set-theoretic models described in the appendices. These models are used to establish the consistency and non-triviality of theories of grounding and granularity formulated in our higher-order language.

Think of propositions as being formed iteratively in an infinite sequence of stages. We begin at stage 0 with one proposition for each set of possible worlds; call these propositions *truth conditions*. At stage n we form all conjunctions and disjunctions of sets of propositions of level less than n, where the *level* of a proposition is the first stage at which it is formed. For example, at stage 1, for every non-empty set of truth conditions X we form a conjunction of all members of X and a disjunction of all members of X. No two sets of propositions have the same conjunction or the same disjunction; no conjunction is identical to any disjunction; and no truth condition is identical to any conjunction or disjunction. Notice that there is no conjunction or disjunction of the set of all propositions: in order for there to be a conjunction and a disjunction of a set of propositions, there must be a stage by which all of its members are formed.[7]

Universal generalizations are identified with conjunctions of their instances and existential generalizations are identified with disjunctions of their instances. In order to ensure that these conjunctions and disjunctions exist, it must be that, for every generalization, there is a stage by which all of its instances are formed. We ensure this as follows. We identify (monadic) properties and (polyadic) relations with functions from entities of the relevant types to propositions. In the case of relations, there is a relation for every such function. In the case of properties things are more complicated: there are not as many properties as functions from entities of the relevant types to propositions. To specify which properties there are, we begin by classifying types into *ranks*: non-monadic types have rank 0, and a type $\langle\tau\rangle$ has rank one greater than the rank of τ. A function from entities of type τ to propositions corresponds to a property if and only if it assigns no entity a proposition whose level exceeds the rank of τ. For example, properties of individuals, of propositions, and of relations are identified with functions from entities of the relevant types to truth conditions; properties of such properties are identified with function from such properties to propositions of level 0 or 1; etc.

This account ensures that, for every property, there is a conjunction and a disjunction of all of its instances (where a proposition is an *instance* of a property just in case it is one of the values of the corresponding function). Moreover, it ensures that the function which maps every monadic property of type $\langle\tau\rangle$ to the conjunction of its instances will itself correspond to a property of type $\langle\langle\tau\rangle\rangle$ (and likewise for disjunction). So not only is there guaranteed to exist, for every property $F^{\langle\tau\rangle}$, a conjunction of all of its instances, which we might call the universal generalization of F, but there is also property $U^{\langle\langle\tau\rangle\rangle}$ such that, for any property $G^{\langle\tau\rangle}$, applying U to G yields the universal generalization of G. We interpret the universal quantifier $\forall_\tau$ as expressing this property. For example, $\forall_{\langle\langle\rangle\rangle}\forall_{\langle\rangle}$ – intuitively, the proposition that every property of propositions applies to all propositions – will be the (false) level-1 proposition whose conjuncts are all and only the conjunctions of level-0 propositions.

Let us now turn to the interpretation of binary conjunction and disjunction – that is, the connectives $\wedge$ and $\vee$. The most straightforward option is to interpret $\wedge$ using the function that maps every ordered pair of proposition to the conjunction whose conjuncts are exactly the members of the pair, and likewise for $\vee$ and disjunction. Since we place no restrictions on which functions from ordered pairs of propositions to propositions count as relations, this interpretation is available. But it may not be the most attractive way of thinking about binary conjunction. This is because it prevents us from using $\wedge$ to characterize the relation of one proposition being a conjunct of another in cases where the latter proposition has more than two conjuncts.

To avoid that expressive limitation, we could instead interpret $\wedge$ as the function that maps every ordered pair of propositions to the conjunction with those two

[7]This idea was inspired by Zeng [49], in which non-basic propositions are formed iteratively from basic, modally individuated ones in such a way that non-basic propositions are individuated by their immediate non-factive full grounds, in the terminology of Fine [15].

propositions as its conjuncts *unless* one of the two propositions is a conjunct of the other, in which case it maps them to the latter proposition. This allows us to define p being a conjunct of q as $p \wedge q = q$, and likewise for disjuncts. (More on the interpretation of $=$ below.) We will then be able to characterize the conjunctive/disjunctive hierarchy of propositions within our higher-order language. Little of what follows depends on which interpretation of $\wedge$ and $\vee$ we adopt; both options are described in the appendix.

Let us now turn to the interpretation of λ-terms. Consider $(\lambda p.p)$. It cannot express the function that maps every proposition to itself, since that function does not correspond to a property of propositions: the only functions that correspond to properties of propositions are functions from propositions to truth conditions. But every proposition straightforwardly determines a truth condition, since a conjunction is true at a world if and only if all of its conjuncts are, a disjunction is true at a world if and only if one of its disjuncts is, and every proposition it is built from truth conditions through the operations of conjunction and disjunction. $(\lambda p.p)$ should therefore express the function that maps every proposition to its truth conditions.

What about other λ-terms? The simplest option is to follow Fritz's suggestion that "the application of properties specified using λ-terms obliterates structural content", yielding mere truth conditions. Since every function from entities of the relevant types to truth conditions corresponds to a property or relation, this interpretation is available. But it may not be the most attractive way of thinking about λ-terms. To see why, note that Fritz motivates his suggestion with the idea that "there is a metaphysical division between logical and non-logical properties, with quantifiers and Boolean connectives being logical, and the properties expressed by λ-terms [...] failing to be logical", where only non-logical properties ever obliterate propositions' structure when predicated of those propositions. And the present picture does not support this idea, since predicating negation of any proposition always yields mere truth conditions (the opposite of the truth conditions of the proposition being negated), despite $\neg$ being a logical constant rather than a λ-term. Whereas Fritz appeals to "a metaphysical division between logical and non-logical properties [and relations]", we appeal to a metaphysical division between (monadic) properties and (polyadic) relations.

Note that while Fritz's diagnosis may seem more natural, it faces a serious problem: it cannot handle a variant of Krämer's puzzle with $\neg$ in place of $(\lambda p.p)$. This is because his claim that Boolean connectives contribute to propositional structure implies that $\exists_{\langle\rangle}\neg$ and $\neg\neg\exists_{\langle\rangle}\neg$ are distinct propositions, with the latter having double negation structure the former lacks. If that is right, then the latter is presumably grounded in the former. Since the latter is also a true instance of the former, this is inconsistent with (7) given the asymmetry of grounding. We will return to the question of whether the differential treatment of monadic properties and polyadic relations is well motivated.

A more conservative approach to λ-terms allows them to behave as much as possible like one would expect them to, obliterating structure only when the functions they would otherwise correspond to are not among the space of properties. This proposal, unlike Fritz's, will validate η-conversion (the intersubstitutability of terms of the form F and $(\lambda x_1 \ldots x_n.Fx_1 \ldots x_n)$), and β-conversion for polyadic λ-terms. While this more conservative approach strikes me as more attractive, nothing in what follows turns on which interpretation of λ-terms we adopt; both options are described in the appendix.

Finally, consider grounding. As discussed above, grounding theorists tend to hold that true generalizations are grounded in their true instances. Having identified generalizations with conjunctions or disjunctions of their instances, this principle follows from the claim that true conjunctions are grounded in their conjuncts and disjunctions are grounded in their true disjuncts. In addition to that claim, I will also assume that grounding is transitive, and that it relates as few propositions as possible consistent with these assumptions (although this assimilation of all grounding relationships to cases of logical grounding is merely for concreteness, and is not essential to the present reply to Krämer's puzzle, as is shown in Appendix G). So p grounds q if and only if there is a chain of (at least two) true propositions starting with p and ending with q such that, for any two consecutive propositions in the chain, the preceding one is either a conjunct or a disjunct of the one after it.

But specifying which propositions ground which others is not yet to specify an interpretation of the grounding connective. To do that, we need to say what function from pairs of propositions to propositions the grounding relation corresponds to. Here is a natural proposal. There are exactly as many ways for p to ground q as there are non-trivial chains of propositions $r_1, \ldots, r_n$ the first of which is p, the last of which is q, and every one of which is a conjunct or disjunct of the one after it. So it is natural to identify the proposition that p grounds q with a disjunction, each disjunct of which is the conjunction of the members of some such chain of propositions linking p and q. If there there are no such chains, this would be the 'disjunction' of the empty set of propositions, which in the present setting we may identify with the 0-level proposition with contradictory truth conditions.

(So far we've focused on the notion of one truth partially grounding another, rather than the perhaps more widespread notion of a *collection* of truths collectively *fully* grounding another. There are a few reasons for this. One is that the issues raised by Krämer's puzzle are more easily stated in terms of partial grounding. Another is that, whereas the ways for p to partially ground q correspond to chains of propositions linking p and q, the ways for a collection of propositions Γ to fully ground p correspond to a much more complicated pattern of propositions. The final reason is that formulating generalizations about full grounding requires expanding our language with devices for generalizing about collections of propositions, which though technically straightforward can be a distracting complication. The details are in Appendix B, along with a treatment of full ground parallel to the present treatment of partial ground and discussion of its implications regarding the grounds of grounding facts.)

The picture just sketched is offered as an informal gloss on the models described in Appendix A. In order to assess whether it offers a response to Krämer's puzzle of the general sort Fritz imagined, we need to investigate the theory of grounding and granularity determined by that class of models. To do this, a few definitions are in order. First, for every type τ, we define a predicate $=_\tau$ as $(\lambda x^\tau y^\tau.\forall F(Fx \leftrightarrow Fy))$. Our theory entails that $=_\tau$ behaves like an identity predicate, in that it is reflexive and licenses the intersubstitution of its flanking terms. Next, we define the operator $\Box$ as $(\lambda q.(\lambda p.p)q = (\lambda p.p)(q \rightarrow q))$. Our theory entails that $\Box$ behaves like a necessity operator: it applies to all and only propositions whose truth conditions are trivial, and obeys the modal logic S5.

The theory validates the transitivity, irreflexivity, and factivity of grounding (if p grounds q, then both are true). It validates the principle (6) above, that existential generalizations are grounded in their true instances, and the principle below, that true universal generalizations are grounded in their instances:

$$\forall F(\forall_\tau F \to \forall x(Fx \prec \forall_\tau F)) \tag{11}$$

Does the theory validate the principle that true conjunctions/disjunctions are grounded in their true conjuncts/disjuncts? This is a more subtle question, turning on which of the two aforementioned interpretations of $\wedge$ and $\vee$ we adopt. The first interpretation, according to which the conjuncts of $p \wedge q$ are exactly p and q (and likewise for disjunction), validates the principles:

$$\forall p \forall q(p \wedge q \to (p \prec p \wedge q)) \tag{12}$$

$$\forall p \forall q(p \to (p \prec p \vee q)) \tag{13}$$

(The same holds for right conjuncts/disjuncts, since conjunction and disjunction are commutative: $p \wedge q = q \wedge p$ and $p \vee q = q \vee p$.) The second interpretation, according to which $p \wedge q = q$ if p is a conjunct of q, and likewise for disjunction, cannot validate these principles given the irreflexivity of grounding. But it does validate the principles:

$$\forall p \forall q((q \wedge Cpq) \to p \prec q) \tag{14}$$

$$\forall p \forall q((p \wedge Dpq) \to p \prec q) \tag{15}$$

Here C (being a conjunct of) and D (being a disjunct of) are defined in the way discussed above (i.e., $C := (\lambda pq.(p \wedge q) = q)$ and $D := (\lambda pq.(p \vee q) = q)$).

As noted in the previous section, Fritz's response requires failures both of β-conversion and of β-grounding. Indeed, the latter principle fails in the following quite general way:

$$\forall p \forall q \neg(p \prec (\lambda p.p)q) \tag{16}$$

This is because predicating $(\lambda p.p)$ reduces propositions to their truth conditions, and truth conditions have no conjunctive or disjunctive structure and hence have no grounds.[8] But although β-conversion fails, it does so only in hyperintensional contexts, since the models validate the following principle:

$$\forall x_1 \ldots \forall x_n \Box(\varphi \leftrightarrow (\lambda x_1 \ldots x_n.\varphi)x_1 \ldots x_n) \tag{17}$$

[8]However, the theory developed here does not vindicate a different suggestion Fritz makes that "Tp grounds a truth p if p is not itself a truth-ascription" (where T abbreviates $(\lambda p.p)$ and is informally glossed as "truth"; alternative accounts of "it is true that" are considered in Sections 5.1 and 6). In support of this suggestion he writes that "if p has structural content, for example in virtue of being a conjunction, then Tp strips away this structural content. The remaining purely qualitative proposition is naturally taken to ground any conjunctive proposition with the same logical content." But I don't think this is a natural suggestion, since it is natural to think that if p is a true conjunction of two modally independent structureless propositions q and r, then the only grounds of p will be q and r, and hence cannot include $(\lambda p.p)p$. Fritz's claim to the contrary strikes me as an overhasty generalization from (8).

Finally, the theory requires no restrictions on classical quantificational logic, validating the following schematic principle of universal instantiation:

$$\forall x\varphi \rightarrow \varphi[a/x] \text{ provided } a \text{ is free for } x \text{ in } \varphi. \tag{18}$$

This is worth noting because it contrasts with a very different style of response to puzzle's like Krämer's, according to which the culprit is quantificational logic itself. For example, following Russell [40], we might deny that we can instantiate a bound variable p with formulas like $\exists pp$ which themselves involve quantification into sentence position, and hence cannot infer (3) from (2). Consideration of such views is beyond the scope of this paper.[9]

Let's take stock. We have seen that the core of Fritz's proposed resolution of Krämer's puzzle can indeed be fleshed out into a "consistent theory of propositional granularity and grounding". And the result has a number of theoretical virtues. The theory is strong (in settling many question of propositional granularity), simple (in doing so systematically), and reasonably parsimonious (since, at least if we interpret conjunction and disjunction connectives judiciously, we can articulate the central aspects of the model construction using corresponding generalizations in our higher-order language). The theory is also logically conservative: it validates classical logic and an unrestricted principle of universal instantiation, and β-conversion fails only in hyperintensional contexts. And it is in many respects ground-theoretically orthodox: true generalizations are grounded in their true instances, and (appropriately understood) true conjunctions and disjunctions are grounded in their true conjuncts and disjuncts.

Indeed, there are many respects in which these models improve on anything currently available in the literature on grounding. While that literature contains many examples of sophisticated models used to interpret formal language containing grounding connectives, these investigations tend to be limited in that the languages in question allow us to embed only a limited range of formulas under grounding connectives. For example, one might be able to talk about the grounds of grounding claims, or about the grounds of conjunctions and disjunctions, but not both, or the grounds of necessities but not of negations. In general, models tend to be purpose-built to characterize the grounds of propositions formed by a particular operation of interest (such as necessitation, conjunction, grounding, etc.). Whether these various modeling strategies can be combined is often unclear, and how they might be extended to languages with other constants and connectives is a matter of guesswork.

[9]Note that (18) is a weaker version of universal instantiation than the one defended by [5] against [3] (although those authors assume β-conversion, in which case the two versions are equivalent). The difference concerns the fact that we are working in a relationally typed language rather than a functionally typed one. In functionally typed languages (which the aforementioned authors use), $(\vee\varphi)$ is itself a term of the same type as $\neg$ (and $\varphi \vee \psi$ abbreviates $(\vee\varphi)\psi$). Having such a term would make $\forall p((\vee\varphi)p \rightarrow ((\vee\varphi)p \prec \exists_\tau(\vee\varphi)))$ a sentence. This sentence would be false, since, for any q, no proposition is grounded in every truth that is the disjunction of some p with q (since it would then ground itself *via* its own disjunction with q), and hence a counterexample to the validity of (6) for $\tau = \langle\rangle$. But (6) is valid in our relationally typed language, which has no term $(\vee\varphi)$; it does have $(\lambda p.\ \vee\varphi p)$, but replacing $(\vee\varphi)$ with $(\lambda p.\ \vee\varphi p)$ turns the above would-be falsehood into a truth.

The basic limitation of such projects isn't that the space of propositions is constructed with particular logical operations in mind. After all, the present theory began with the idea that the space of propositions can be understood as built up from truth conditions by iterating two operations of conjunction and disjunction. The issue is rather that the models one finds in the literature have nothing corresponding to domains of properties and relations, or any general way of thinking about the available interpretations for new constants and connectives we might add to our language. By contrast, the present construction shows how a theory of grounding and propositional granularity can be embedded in a general theory of the granularity of propositions, properties and relations. This makes the present theory robust in a way that existing theories of grounding in the literature are not: we know how to conservatively extend the theory with new constants and connectives, whereas most extant theories of grounding require adding new structure to their models whenever a new connective is added to the language. In fact, while each of the following notions has been modeled separately in the grounding literature, I know of no prior theories of grounding that allow one to simultaneously model the grounds of conjunctions, disjunctions, generalizations, negations, necessitations, and grounding claims.[10]

But we have not yet answered Fritz's question. He asked "whether developing the present picture *in natural ways* will actually lead to a consistent theory of propositional granularity and grounding" (emphasis added). And while the present theory may be strong, simple, and parsimonious, there are a number of respects where it is nonetheless unnatural. Some of these respects are mere artifacts of the model construction, as we will see in the Section 3. Others arise from the very features of the theory that allows it to respond to Krämer's puzzle, as we will see in Section 5.

3 Refinements

This section considers two objections to the theory just developed, one to do with negated propositions and the other to do with ungrounded propositions. It then describes a modification of the theory that responds to these worries. The picture that emerges is a kind of logical atomism.

Grounding facts are often thought to correspond to true answers to "why"-questions, at least on an appropriately metaphysically loaded understanding of those questions. But at least in ordinary conversation, "why not"-questions are not much less common than "why"-questions. So if "why"-questions point to a rich subject matter of what grounds true propositions, we might expect "why not"-questions to point to a rich subject matter of what ground the negations of false propositions. But on the present proposal, there is no such subject matter. For negation is a property of propositions, and so predicating it yields mere truth conditions with no grounds at all.

[10]On the latter, Litland [33] writes: "The existing solutions to puzzles [of ground] have all been worked out for languages without grounding operators in the object language. Once we introduce grounding operators into the object language, the constructions have to be redone, taking into account one's favorite proposal about grounding ground. This is not a trivial exercise, since principles about grounding ground can generate new puzzles of ground."

This prediction may not be as problematic as it at first looks. This is because it is open to grounding theorists to deny that explaining the falsity of false propositions amounts to explaining the truth of those propositions' negations. They might hold, instead, that it amounts to explaining the truth of those propositions' *involutions*. Intuitively, two propositions are involutions of each other just in case they are constructed in the opposite way from the opposite ingredients. Where p is ultimately built from some truth conditions by forming conjunctions and disjunctions in some pattern, its involution is the proposition built from the opposite truth conditions by forming disjunctions instead of conjunctions and conjunctions instead of disjunctions. So the involution of p has the same complexity as p but the same truth conditions as $\neg p$. Our proposal, then, is that, where q^* is the involution of q, we say that p *falsifies* q just in case p grounds q^*, and hold that "why-not" questions, appropriately understood, point to a rich subject-matter of what falsifies false propositions.[11] The proposition p *falsifies* q implies that p is true and q is false, but it is not the proposition that p grounds $\neg q$. Rather, it is the proposition that p grounds q^*, where q^* is the *involution* of q. In this way there is just as much structure to falsification as there is to grounding, and the distinction between falsifying q and grounding $\neg q$ is no different than the equally crucial distinction between grounding q and grounding $(\lambda p.p)q$.

While it may be somewhat inelegant to have two different connectives, one for grounding and another for falsification, this is not so great a cost. But there is a deeper worry about the way that predicating properties of propositions inevitably dissolves them down to their truth conditions. The worry is simply that, on a picture of propositions as having a hierarchical conjunctive and disjunctive structure, we should not think that for every truth condition there is a corresponding ungrounded proposition.

We can sharpen the worry by showing how it conflicts with the following picture, reminiscent of Wittgenstein's logical atomism. According to this picture, ungrounded truths are special: they are *basic*, and as such they are modally independent of one another. The present theory is grossly incompatible with this vision. For example, for any two modally independent ungrounded propositions, there is a third ungrounded proposition that is necessarily equivalent to their conjunction. This is because, for any proposition q, $(\lambda p.p)q$ is both ungrounded and necessarily equivalent to q.

One might think that this prediction, however unwelcome, is an inevitable feature of the kind of theory we have been developing. For the demand that predications of $(\lambda p.p)$ always be level-0 propositions was imposed to ensure that there is a disjunction of all such predications and a property, namely existential generality, that, when predicated of $(\lambda p.p)$, yields this disjunction. But it turns out we can ensure this without supposing (as we did in the previous section) that a function from entities of type τ to propositions corresponds to a property of such entities only if the function doesn't assign any entity a proposition whose level exceeds the rank of τ. It is enough to suppose that a function from entities of type τ to propositions corresponds to a property only if the function doesn't assign any entity a proposition whose level

[11] Litland [35] sympathetically explores a version of this idea.

exceeds $s(r)$, where r is the rank of τ and s is some strictly increasing function on natural numbers. A natural implementation of this more general idea is the following: a function from entities of type τ to propositions determines a property just in case no entity is assigned a proposition whose level *exceeds by more than 2* the rank of τ.

To see why this is a natural proposal, we need to modify our picture of propositions. We begin at stage 0 with a collection of logical atoms. The negation of every logical atom is a logical atom, and propositions are formed iteratively as conjunctions and disjunctions of logical atoms. Unlike before, we now allow a conjunction of the empty set of propositions, which we think of as a trivial tautology (having only true conjuncts), and a disjunction of the empty set of propositions, which we think of as a trivial contradiction (having no true disjuncts). Possible worlds now correspond to maximally consistent sets of logical atoms, sets which contain every level-0 proposition or its negation but never both. As before, the truth conditions of a proposition are the set of worlds in which it is true.

We can no longer demand that predicating $(\lambda p.p)$ always yields a level-0 proposition. For $(\lambda p.p)q$ must have the same truth conditions as q, and not every proposition has the same truth conditions as any level-0 proposition (or of any level-1 proposition). But once we reach level 2 all truth conditions are represented. Moreover, there are natural ways of taking any proposition whose level is greater than 2 and flattening it down to a level-2 proposition, since every nesting of conjunctions and disjunctions can be reduced to a conjunctive/disjunctive normal form (i.e., to either a conjunction of disjunctions of level-0 propositions, or a disjunction of conjunctions of level-0 propositions).[12] Instead of having $(\lambda p.p)$ dissolve propositions down to mere truth conditions, as we did above taking our cue from Fritz, we can instead have it do something less drastic, namely flattening propositions of level greater than 2 down to level-2 normal forms, and leaving all other propositions alone. Similarly, negation maps every proposition of level greater than 2 to the normal form of its involution. The details are given in Appendix C.

Note that our modified theory no longer allows us to define a notion of necessity in terms of having the same truth conditions as a tautology, since we cannot talk about the truth conditions of a proposition by predicating $(\lambda p.p)$ of it, or indeed by predicating anything else, since truth conditions are not propositions. There may yet be a more complicated way of defining in purely logical terms what it is for a proposition

[12]There are different ways in which this might go. For example, flattening propositions might be conceived as crushing them, breaking their large scale structure, and fusing together their ultimate particles, as in the formation of sedimentary rocks. Or it might be conceived as preserving their shape, but putting enough pressure on them to change the organization of their elementary constituents, as in the formation of metamorphic rocks. Cashing out this metaphor, consider this conjunction of disjunctions of conjunctions of logical atoms: $\bigwedge\{\bigvee\{\bigwedge\{a, b\}, \bigwedge\{c, d\}\}, \bigvee\{\bigwedge\{e, f\}, \bigwedge\{g, h\}\}\}$. Sedimentary flattening changes its large scale structure, turning it into a disjunction, and fuses the ultimate conjuncts together into bigger conjunctions, yielding $\bigvee\{\bigwedge\{a, b, e, f\}, \bigwedge\{a, b, g, h\}, \bigwedge\{c, d, e, f\}, \bigwedge\{c, d, g, h\}\}$. Metamorphic flattening, by contrast, keeps the overall conjunctive structure, but redistributes the members of the original proposition's ultimate conjuncts among the disjuncts, yielding $\bigwedge\{\bigvee\{a, c\}, \bigvee\{a, d\}, \bigvee\{b, c\}, \bigvee\{b, d\}, \bigvee\{e, g\}, \bigvee\{e, h\}, \bigvee\{f, g\}, \bigvee\{f, h\}\}$. The notion of flattening defined in the appendix is metamorphic.

to have trivial truth conditions.[13] Or we might introduce necessity as a primitive logical constant, perhaps interpreted by the function that maps every proposition with trivial truth conditions to the trivial tautology, and every proposition without trivial truth conditions to the trivial contradiction. This treatment will again yield a standard S5 modal logic, with truth necessity claims not being (partially) grounded in any propositions.

4 Granularity First

Our investigation so far has taken a *granularity first* approach to the theory of grounding.[14] Here is what I mean. We haven't spent much time questioning what we want from a theory of grounding, at least *vis-à-vis* conjunction, disjunction, and generalization. Insofar as possible, we want conjuncts to ground true conjunctions, true disjuncts to ground disjunctions, and true instances to ground true generalizations, without threatening the transitivity and asymmetry of grounding. But to what extent *is* this possible? This is a very hard question in light of impossibility results like Krämer's. To make progress, we need to explore different theories of propositions' structure and the extent to which they allow us to validate the principles of logical grounding on our wishlist while respecting the structural constraints on grounding. This section discusses a new impossibility result that highlights important constraints on such theorizing.

As Fritz [19] notes, most theorizing about grounding freely deploys notions of one proposition being a (binary) conjunct of another and of one proposition being an instance of another, which are taken to obey the principles:

$$conj(p, q \wedge r) \leftrightarrow (p = q \vee p = r) \tag{19}$$

$$inst(p, \forall q\varphi) \leftrightarrow \exists q(p = \varphi) \tag{20}$$

Surprisingly, Fritz [18] shows that these two seemingly innocuous principles are in fact jointly inconsistent.[15] This result is notable for two reasons. First, it differs from other limitative results about propositional granularity like the Russell-Myhill theorem (discussed in Section 6) in that each of the two principles is separately consistent.

[13]This is possible, for example, if we adopt the interpretation of $\wedge$ and $\vee$ that allows us to talk about propositions' conjunctive and disjunctive structure within our higher-order language.

[14]Dorr [12] is a manifesto for granularity-first approaches in metaphysics more generally. I am sympathetic, but the present methodological point is much weaker: a granularity-first approach to logical grounding is urgent in light of the paradoxes of grounding, as Fritz [19] forcefully argues, whatever one thinks about such approaches in other areas; see also [8]. Krämer [26, 27] is a fellow traveler; his approach is to first determine how fine-grained propositions are as far as grounding is concerned, although he is open to propositions being fine-grained in further ways that make no difference to grounding.

[15]*Proof*: Let $B(X) := \forall p(p \wedge (p \vee Xp))$. For any X, if $\forall p(Xp \rightarrow \neg p)$, then (19) and (20) imply that, for all p, Xp if and only if p is a false conjunct of an instance of $B(X)$ that also has a true conjunct. Since $B(X)$ is false for any X, (19) and (20) imply that B gives an injection from collections of false propositions to false propositions, in violation of Cantor's theorem.

(The consistency of (20) is not straightforward and is established in Appendix E.) Second, it shows that the view of propositional granularity developed in the previous two sections can be motivated independently of any considerations about grounding. It can be motivated directly by a desire to have a theory that supports both a robust theory of conjuncts and of instances. By interpreting "is a (binary) conjunct of" in our models as being a (model-theoretic) conjunct of a conjunction with no more than two conjuncts, and interpreting "is an instance of" in our models as being a (model-theoretic) conjunct or disjunct of, we validate both (19) and

$$inst(p, \forall_{\langle\rangle} F) \leftrightarrow \exists q(p = Fq) \tag{21}$$

The the fact that we validate (19) and (21), despite the inconsistency of (19) and (20), is exactly parallel to the fact that we validate (5) and (7) despite the inconsistency of (5) and (1). In both cases, the tenability of these principles depends on distinguishing $(\lambda p.p)\varphi$ and φ: when these propositions differ, the former has less structure than the latter despite being expressed by a more complex formula.

These reflections have important ramifications for the main extant approaches to the logical puzzles of grounding in the literature, first advanced by Fine [14] and later refined by Litland [31, 34] and others. These accounts build on ideas about grounding as a relation between *sentences* in Kripke's ([29]) theory of truth, and transposing these ideas from sentences to propositions. While these theories are complex and varied, we can get a feel for their relevant features by considering how they respond to puzzles like the one with which we began. These theories hold that the proposition that something is true grounds the proposition that the proposition that something is true is true, and not *vice versa*. They thus deny that true generalizations are always grounded in their true instances. This denial is motivated (very roughly) by the idea that all logically complex truths need grounds, the (true) proposition that the proposition that something is true is true can be grounded only via the proposition that something is true, and grounding is asymmetric, so it does not also go the other way; and this is okay, since the proposition that something is true has other true instances to ground it without generating circles of grounding (e.g., the proposition that the proposition that grass is green is true).[16] While these views are articulated in the context of first-order theories of propositions, the parallel response in the present setting is clear: $\exists_{\langle\rangle}(\lambda p.p) \prec (\lambda p.p)\exists_{\langle\rangle}(\lambda p.p)$, and not *vice versa*, despite the latter being an instance of the former. These views therefore make the opposite prediction to the theory we have been exploring.

I think Fritz's impossibility result provides a strong reason to be skeptical about these Kripke-inspired approaches to the logical puzzles of grounding. Just as one sentence being a conjunct or an instance of another are notions that are central to Kripkean grounding relations between sentences, these approaches presuppose that parallel notions make sense in application to propositions. But if this is understood in the most straightforward way, as in as (19) and (20), then the resulting theory is inconsistent (given classical logic, universal instantiation, etc.). And if it is

[16] Correia [6], Lovett [36] and Woods [48] argue that we should instead deny that grounding is asymmetric; see Litland [31, Section 6.8] for a reply.

understood in a more ecumenical way, for example as (19) and (21), then this involves a disconnect between the structure of propositions and the structure of the sentences expressing them. The predictions of the approach are then unclear, since they can no longer be read off of parallel theories of grounding understood as a relation between sentences. To see what they predict, the Kripkean approaches must be paired with some concrete picture of propositional granularity. Yet the only theory of granularity we have seen so far in which (19) and (21) hold is one where grounding is most naturally interpreted in a way where true generalizations *are* always grounded in their true instances, which is what the Kripkean response denies. I do not claim to have shown that there is no well-motivated consistent theory of the granularity of propositions, properties and relations within which a broadly Kripkean response to Krämer's puzzle is both formulable and attractive. But I am not aware of any such theory.[17]

So I am pessimistic about the framework that has led many authors to prioritize the principle that any truth grounds the fact that it is true over the principle that any truth grounds any true generalization of which it is an instance. But I do not want to dismiss the idea that we should uphold the former principle rather than the latter, and that the theory developed in the previous two sections should therefore be rejected. The next section explores the costs of denying that every truth grounds the fact that it is true, and some related tensions in the view developed above. The following section then develops a different theory of grounding and granularity that responds to these tradeoffs in the opposite way.

5 Two Objections

In this section I'll present two related objections to the theory developed in Sections 2 and 3. The objections turn on the fact that the theory requires fundamentally different treatments of monadic properties and dyadic relations. In brief, no monadic operator can have the kind of ground theoretic behavior that conjunction has, and no quantifier applying to relations can have the kind of ground theoretic behavior that the existential quantifier has. On their face, these look like invidious distinctions. To the extent that they are, they constitute an objectionable prediction of

[17] This is not the end of the matter. Kripke-inspired theories may still be well-motivated as solutions to puzzles about the grounds of claims about sentential truth, and hence give an independent reason to deny that true generalizations are always grounded in their true instances – i.e., to deny that the fact that "there are true sentences" is a true sentence grounds the fact that there are true sentences. I am not convinced that such theories are well motivated though. They seem to tacitly rely on some disquotational grounding principle. Could it be: "φ" is true $\rightarrow$ $\varphi \prec$ "φ" is true? I don't think so, since that principle classically implies the inconsistent T-schema given a classical semantics for negation (i.e., that a sentence s is true if and only if $\ulcorner \neg s \urcorner$ is not true). What about the following more cautious principle linking Kripkean sentential grounding and metaphysical grounding: if "φ" and "ψ" are both true and the former Kripke-grounds the latter, then $\varphi \prec \psi$? This principle is still strong enough to derive the alleged grounding claim about "there are true sentences", without implying the T-schema. However, I am not sure how well-motivated the principle is once we reject the T-schema and other classically inconsistent disquotational principles, as I believe we should; see [2].

the theory that is not a mere accident of the particular way we have chosen to develop it. However, it turns out these predictions can be avoided if we are willing to radically rethink the nature of conjunction and disjunction, by adopting a syncategorematic treatment of Boolean connectives.

5.1 Truth

We saw in Section 1 that Fritz's response to Krämer's puzzle requires denying that $(\lambda p.p)\exists(\lambda p.p)$ is either the same proposition as or grounded in $\exists(\lambda p.p)$. But the argument was quite general. There can be no property of propositions T that both applies to every truth and is such that, for all true p, Tp is either identical to or grounded in p.

One might argue that it is unprincipled to deny the existence of such a property if we are willing to accept that true conjunctions are grounded in their conjuncts. For presumably the ground-theoretic behavior of $\wedge$ isn't unique to binary conjunction. Surely we can introduce a similarly behaved notion of ternary conjunction $\wedge_3^{\langle\langle\rangle,\langle\rangle,\langle\rangle\rangle}$ whose ground-theoretic behavior is analogous to that of $\wedge$, and likewise for a four-place connective $\wedge_4^{\langle\langle\rangle,\langle\rangle,\langle\rangle,\langle\rangle\rangle}$, and so on. But then shouldn't we also be able to introduce an analogous monadic connective $\wedge_1^{\langle\langle\rangle\rangle}$? Those who accept Fritz's response must deny that this is possible. The objection is simply that such a denial is implausible.

Here is a possible rejoinder in defense of the theory of grounding we have been exploring. The above objection holds that we should be able to understand a notion of monadic conjunction by analogy with an infinite family of binary, tertiary, and higher-arity conjunction connectives. But how does this work? The proposed notion $\wedge_1$ is not meant to be *definable* as self-conjunction, i.e. as $(\lambda p.p \wedge p)$. For the demand that $(\lambda p.p \wedge p)$ have the relevant grounding behavior is no more obvious than that $(\lambda p.p)$ should. Moreover, unlike notions of n-ary conjunction for $n \geq 2$, we cannot see that it should be a kind of *conjunction* simply by reflecting on the relevant one-column truth table, since that truth table doesn't distinguish the proposed notion of monadic conjunction from a parallel notion of monadic disjunction. And according to the present picture of propositions, this is a substantive distinction, since the proposition with p as its sole conjunct is not the same as the proposition with p as its sole disjunct.

Is this rejoinder compelling? The dialectical situation is delicate. There is of course a coherent position here for the grounding theorist, as the model constructions in the appendix show. And the idea that the monadic case is special because there is nothing in the corresponding truth table to tell apart conjunction and disjunction is an interesting strategy for trying to make the position seem more principled. Note, however, that Fritz's response to Krämer's puzzle does not force us to distinguish between the conjunction with p as its lone conjunct and the disjunction with p as its lone disjunct. It would be straightforward to modify the model construction to avoid drawing such a distinction, holding instead that, in the monadic case, there are simply "junctions" – propositions with a single "junct" that are truth conditionally equivalent to it and, if true, are grounded in it. Indeed, those attracted to the idea that the structure of grounding relations (on a non-factive understanding of grounding) is what individuates propositions will be suspicious of distinguishing singleton conjunctions and singleton disjunctions, and so should be wary of relying on such a distinction to defend their position.[18]

On the other hand, singling out the monadic for special treatment is nowhere near as *ad hoc* as, say, singling out binary connectives, or some less natural collection of types. While it is not something we would have expected in advance of inquiry, questions of granularity are an area of metaphysics where surprises are to be expected. So while the special treatment of monadic properties may not be a selling point of the present framework, it isn't clearly a fatal flaw. I don't think it is *obvious* that "it is true that . . . , or some notion of 'monadic conjunction', must to have the prohibited grounding behavior.

5.2 Relations

We just saw that, if existential generalizations are grounded in their true instances, then there is no monadic counterpart of conjunction – no property of propositions that, when applied to a true proposition, yields a truth that is grounded in that proposition. I will now establish a parallel result in the opposite direction: if true conjunctions/disjunctions are grounded in their true conjuncts/disjuncts, then there is no counterpart of the existential propositional quantifier applying to relations between propositions – no property that, when applied to a relation that some proposition bears to another, yields a truth that is grounded in those propositions' being so-related. This presents a significant challenge to the view, since the idea that true predications ought to ground existential generalizations of what is predicated seems to be no less compelling when predicating polyadic relations than when predicating monadic properties.

Consider an example. Intuitively, the fact that Caesar was killed grounds the fact that someone was killed, and this is predicted by (6). By the same token, it seems that the fact that Brutus killed Caesar should ground the fact that someone killed someone. But this is not predicted by (6). That principle predicts that the fact that Brutus killed someone grounds the fact that someone killed someone – and, more generally, that $(\lambda x.\exists y Rxy)a$ grounds $\exists x \exists y Rxy$. But it is completely silent about when propositions are grounded by true predications of polyadic relations.

There are two ways in which we might generalize (6) to cover this case. The most straightforward would be the principle:

$$\forall R^{\langle \tau_1, \tau_2 \rangle} \forall x^{\tau_1} \forall y^{\tau_2} (Rxy \rightarrow (Rxy \prec \exists x \exists y Rxy)) \tag{22}$$

But this is not the most natural strategy in the present setting. Intuitively, the instances of the existential generalization of a dyadic relation should be all and only the predications of that relation, just as the instances of the existential generalization of a monadic property should be all and only the predications of that property. Since

[18] Zeng [49] defends a view of this sort; see also Litland [35].

$\exists x \exists y Rxy$ is shorthand for the existential generalization of the property $(\lambda x.\exists y Rxy)$, and predications of this property need not be predications of R (or *vice versa*), we have good reason to consider alternative ways of theorizing about the existential generalizations of binary relations.

One alternative is to introduce primitive existential quantifiers $\exists_{\tau_1,\tau_2}$ of type $\langle\langle\tau_1, \tau_2\rangle\rangle$, which we might think of as ranging over pairs of entities, the first of type τ_1 and the second of type τ_2. $\exists_{\tau_1,\tau_2} R$ will have the same truth conditions as $\exists x^{\tau_1} \exists y^{\tau_2} Rxy$. And just as we can think of $\exists_e F^{\langle e\rangle}$ as the existential generalization of the property of individuals F, we can likewise think of $\exists_{e,e} R^{\langle e,e\rangle}$ as the existential generalization of the relation between individuals R. The idea that true predications of binary relations ground the existential generalizations of those relations can then be formalized as follows:

$$\forall R^{\langle\tau_1,\tau_2\rangle} \forall x^{\tau_1} \forall y^{\tau_2} (Rxy \to (Rxy \prec \exists_{\tau_1,\tau_2} R)) \tag{23}$$

I will now show that (23) is inconsistent with true disjunctions always being grounded in their true disjuncts. First, consider the instance of (23) concerning relations between propositions:

$$\forall R^{\langle\langle\rangle,\langle\rangle\rangle} \forall p^{\langle\rangle} \forall q^{\langle\rangle} (Rpq \to (Rpq \prec \exists_{\langle\langle\rangle,\langle\rangle\rangle} R)) \tag{24}$$

Now recall that $\vee$ is a predicate of type $\langle\langle\rangle, \langle\rangle\rangle$ (and that $p \vee q$ is shorthand for $\vee pq$). So we can instantiate R with $\vee$, p with $\exists_{\langle\langle\rangle,\langle\rangle\rangle}\vee$, and q with an arbitrary falsehood φ, yielding:

$$\vee(\exists_{\langle\langle\rangle,\langle\rangle\rangle}\vee)\varphi \to \vee(\exists_{\langle\langle\rangle,\langle\rangle\rangle}\vee)\varphi \prec \exists_{\langle\langle\rangle,\langle\rangle\rangle}\vee \tag{25}$$

The antecedent is true, since its first disjunct is the true claim that the disjunction relation is instantiated (which it is). So by modus ponens we have:

$$\vee(\exists_{\langle\langle\rangle,\langle\rangle\rangle}\vee)\varphi \prec \exists_{\langle\langle\rangle,\langle\rangle\rangle}\vee \tag{26}$$

But the principle that true disjunctions are grounded in their true disjuncts (in either of the two versions discussed in Section 2) implies the reverse:

$$\exists_{\langle\langle\rangle,\langle\rangle\rangle}\vee \prec \vee(\exists_{\langle\langle\rangle,\langle\rangle\rangle}\vee)\varphi \tag{27}$$

This contradicts the asymmetry of grounding.

This argument doesn't depend on using $\exists_{\langle\rangle,\langle\rangle}\vee$ rather than $\exists p \exists q (p \vee q)$ to regiment the existential generalization of disjunction. It is enough to assume that this claim, however it is understood, would be a proposition that each true disjunction partially grounds. The argument then shows that there can be no such proposition. For such a proposition will be true (by the factivity of grounding), as will the result of disjoining it with any other proposition. Assuming true binary disjunctions with a false disjunct are grounded in their true disjunct, this conflicts with the asymmetry of grounding.

5.3 The Syncategorematic Strategy

I want now to consider a somewhat radical but I think intriguing and instructive possible response to this last argument. That argument relied on disjunction being a relation between propositions. More precisely, it relied on instantiating the variable R with the connective $\vee$ to move from the universal generalization (24) to its instance (25). The response I want to consider rejects this application of universal instantiation, on the grounds that disjunction is not a relation.

The idea is that, fundamentally, conjunction and disjunction are not *relations between* propositions but *operations on* collections of propositions.[19] On this picture, a more metaphysically perspicuous notation would represent these operations not using binary connectives, but instead by special syntactic operations, so that, for any finite set of formulas Γ, we can form two new formulas $\bigwedge \Gamma$ and $\bigvee \Gamma$. Crucially, $\vee$ is no longer a term of our language. This means that (25) is now ill-formed, and so cannot be a counterexample to (24).

Now $(\lambda pq.\bigvee\{p,q\})$ *is* a term, and the view under consideration will accept the corresponding instance of (24):

$$\begin{aligned}&(\lambda pq.\bigvee\{p,q\})(\exists_{\langle\rangle,\langle\rangle}(\lambda pq.\bigvee\{p,q\}))\varphi \rightarrow\\ &(\lambda pq.\bigvee\{p,q\})(\exists_{\langle\rangle,\langle\rangle}(\lambda pq.\bigvee\{p,q\}))\varphi \prec \exists_{\langle\rangle,\langle\rangle}(\lambda pq.\bigvee\{p,q\})\end{aligned} \tag{28}$$

But just as all versions of Fritz's response hold that $(\lambda p.p)\varphi$ can ground propositions that φ does not (e.g., when φ is $\exists_{\langle\rangle}(\lambda p.p)$), the present proposal holds that $(\lambda pq.\bigvee\{p,q\})\varphi\psi$ grounds propositions that $\bigvee\{\varphi,\psi\}$ does not (e.g., when φ is $\exists_{\langle\rangle,\langle\rangle}(\lambda pq.\bigvee\{p,q\})$). This is because disjunctions never ground their disjuncts, so the proposal is committed to rejecting the analogue of (25):

$$\begin{aligned}&\bigvee\{\exists_{\langle\rangle,\langle\rangle}(\lambda pq.\bigvee\{p,q\}),\varphi\} \rightarrow\\ &\bigvee\{\exists_{\langle\rangle,\langle\rangle}(\lambda pq.\bigvee\{p,q\}),\varphi\} \prec \exists_{\langle\rangle,\langle\rangle}(\lambda pq.\bigvee\{p,q\})\end{aligned} \tag{29}$$

The response we are considering does not merely block the argument in Section 5.2. It also allows us to formulate polyadic extensions of the idea that generalizations are grounded in their instances, which can subsume (6) and (11) as well as (23) as special cases. For every sequence of types $\tau_1,\ldots,\tau_n$ we introduce primitive universal and existential quantifiers $\forall_{\tau_1,\ldots,\tau_n}$ and $\exists_{\tau_1,\ldots,\tau_n}$ of type $\langle\langle\tau_1,\ldots,\tau_n\rangle\rangle$. We can then formulate the following two principles:

$$\forall R^{\langle\tau_1,\ldots,\tau_n\rangle}\forall x_1\ldots\forall x_n(Rx_1\ldots x_n \rightarrow (Rx_1\ldots x_n \prec \exists_{\tau_1,\ldots,\tau_n}R)) \tag{30}$$

$$\forall R^{\langle\tau_1,\ldots,\tau_n\rangle}(\forall_{\tau_1,\ldots,\tau_n}R \rightarrow \forall x_1\ldots\forall x_n(Rx_1\ldots x_n \prec \forall_{\tau_1,\ldots,\tau_n}R)) \tag{31}$$

It is straightforward to modify the models in Appendix C to establish the consistency of the resulting theory, in a formal language where conjunction and disjunction are treated not as binary sentential connectives but as operations on sets of sentences. The resulting theory also avoids differential treatment of monadic properties and polyadic relations of the sort discussed in Section 5.1 – relations are now treated in the same way that monadic properties had previously been. The details are in Appendix D.

The resulting theory is not without its costs. For example, there is no way to define the notion of one proposition being a conjunct or disjunct of another, and hence no

[19] Stalnaker [42, 43] develops a syncategorematic account of logical connectives for somewhat related reasons: on his view, no relations between propositions obey the principles governing Boolean connectives, since connectives are not existence-entailing in modal contexts.

way to even uniquely characterize (let alone define) the new quantifiers like $\exists_{\langle\rangle,\langle\rangle}$ in terms of the standard higher-order quantifiers that apply to monadic properties. This might be seen as a cost in terms of parsimony.

A different misgiving about the proposal is that it fails to generalize to related puzzles. Consider the view that "P is a partial ground for knowledge that P", which Fine [15, p.53] attributes to Williamson [46]. In symbols:

$$\forall x^e \forall p(Kxp \to p \prec Kxp) \tag{32}$$

But now consider the existential generalization of knowledge: $\exists_{e,\langle\rangle} K$. This is not merely true but *known*: $\exists x Kx(\exists_{e,\langle\rangle} K)$. It follows that (23) is inconsistent with (32) given the asymmetry of grounding, by an argument parallel to that of the previous section.[20] But the kind of response suggested above about disjunction is much less plausible in the case of knowledge. The response would be that "knows that" should be regimented not as a relation of type $\langle e, \langle\rangle\rangle$ but instead as a basic syntactic operation for forming a formula from a singular term and a formula. This is much less plausible than a syncategorematic treatment of conjunction and disjunction.

This last argument is robust with respect to different views about how what is known grounds our knowledge. For example, Goldman [20, p.101] defends the view that "If S knows that p, then p is a prominent part of the explanation for his believing that p." We might naturally formalize this idea as a generalization about the grounds of our believing the things we know:

$$\forall x \forall p(Kxp \to p \prec Bxp) \tag{33}$$

This principle must be rejected too. I know that someone believes something. So the existential generalization of belief is known: $\exists x Kx(\exists_{e,\langle\rangle} B)$. Equations (23) and (33) are then inconsistent with the asymmetry of grounding.

Or consider a neo-Davidsonian account of the logical form of propositional attitude ascriptions, so that "x knows that p" has the logical form: $\exists e^e(\text{has}(x, e) \wedge \text{that}(e, p) \wedge \text{know}(e))$. A proponent of this view might still think that, if one has a mental state with content that p which amounts to knowledge, then their having that mental state is partially grounded in p:

$$\forall x \forall p \forall e((\text{has}(x, e) \wedge \text{that}(e, p) \wedge \text{know}(e)) \to p \prec \text{has}(x, e)) \tag{34}$$

But now consider the existential generalization of the relation of having a mental state: $\exists_{e,e}\text{has}$. This fact is known: $\exists x \exists e(\text{has}(x, e) \wedge \text{that}(e, \exists_{e,e}\text{has}) \wedge \text{know}(e))$. Equations (23) and (34) are then again inconsistent with the asymmetry of grounding.

Grounding theorists are of course free to reject these principles about knowledge. But it is uncomfortable that they are forced to do so. They could try to assimilate this rejection to the bullet they must already bite about truth, by holding that, when p is known, this knowledge is grounded not in p but rather in the *truth* of p – i.e., in Tp, where T regiments "it is true that ...". But the mere precedent of cases where Tp grounds propositions that p does not (e.g., when $p = \exists_{\langle\rangle} T$) does not make this

[20] See Peels [38] for a version of this argument.

response plausible. To the contrary: the more p becomes alienated from Tp in its grounding behavior, the less plausible it is that the latter as opposed to the former is the appropriate ground of our knowledge of the former (cf. [36, note 14]).

In sum: Perhaps the most promising direction for developing Fritz's solution to Krämer's puzzle involves denying that conjunction and disjunction should be thought of as relations between propositions and denying that states of knowledge are grounded in what is known. While such views are certainly worth exploring, they are also is sufficiently revisionary that alternative solutions are clearly worth exploring too. The next section considers one such alternative.

6 Predicational Structure

All of the theories explored so far have essentially relied on the following convenient assumption about propositional granularity: that universal generalizations have their instances as conjuncts and existential generalizations have their instances as disjuncts. This assumption has let us reduce the theory of grounding generalizations to the theory of grounding conjunctions and disjunctions. But despite its convenience, there are well-known reasons to reject it.

One familiar objection appeals to the claim that some individuals only contingently exists (where for present purposes 'exist' means $(\lambda x.\exists y(y = x))$). Since it is necessary that everything exists ($\Box\forall x\exists y(y = x)$), and every conjunct of a necessary truth is necessarily true, it follows that, if anything exists contingently, then the fact that it exists is not a conjunct of the fact that everything exists.[21] Another familiar objection goes like this. Suppose that the result of predicating a property of an individual is about that individual, that a disjunction is about any individual that any of its disjuncts is about, and that an existential generalization is about only the individuals the property being generalized is about. Provided that not every property is about every individual, it follows that not every instance of an existential generalization is a disjunct of it.[22]

A natural reaction to these considerations is to explore views that align the structure of universal and existential generalizations more closely with the structure of the sentences that express them. For example, many philosophers are attracted to the following principle about monadic predications (which, in our higher-order language, include universal and existential generalizations):

$$\forall F^{\langle\tau\rangle}\forall G^{\langle\tau\rangle}\forall x^{\tau}\forall y^{\tau}(Fx = Gy \to F = G \wedge x = y) \tag{35}$$

In words: when you predicate a property of an entity, which property and entity these were can be recovered from the resulting proposition. If this principle held, it would support a more naïve notion of one proposition being an instance of another: p is an

[21] For arguments that nothing contingently exists see Williamson [47] and Goodman [21].

[22] Correia and Skiles [9, p. 662] defend the claim that instances ground existential generalization by being disjuncts of them, in response to a similar worry.

instance of q just in case p predicates a property and q predicates either universal or existential generality of that property.

Making this informal idea precise raises some subtle issues to do with monadic properties coming in infinitely many types. But we can sidestep these complexities, since it turns out that (35) is classically inconsistent given (18) and the following very weak assumption about applications of λ-terms:

$$(\lambda x_1 \dots x_n.\varphi)a_1 \dots a_n \leftrightarrow \varphi[a_i/x_i]. \tag{36}$$

This result is sometimes known as the Russell-Myhill paradox.[23] It shows that propositions cannot have as much structure as the naïve understanding of one proposition being an instance of another seems to presuppose.

The most common reactions to this result are to either question the basic logical principles needed to derive it or to despair about propositions having any kind of predicational structure of the sort that (35) attempts to encode.[24] But I want to explore a third reaction. It turns out that there are interesting consistent weakenings of (35), which suggest new pictures of propositional granularity that in turn suggest new responses to Krämer's and related grounding paradoxes. I will now outline such a theory of propositional granularity, focussing on the theory of grounding it supports and how it compares with theories that identify being an instance with being either a conjunct or a disjunct.

The simplest version of the Russell-Myhill argument establishes the inconsistency of the instance of (35) for $\tau = \langle\rangle$:[25]

$$\forall F^{\langle\langle\rangle\rangle} \forall G^{\langle\langle\rangle\rangle} \forall p^{\langle\rangle} \forall q^{\langle\rangle} (Fp = Gq \rightarrow F = G \wedge p = q) \tag{37}$$

But (35) is not inconsistent for all types. In fact, it is inconsistent only for types that involve the type of propositions. If we consider only individuals, properties of and relations between individuals, properties of and relations between such entities, and so on, (35) and its polyadic generalization are consistent.

More precisely, let the set of *recoverable* types be the smallest set of types that contains e and that is closed under the formation of *non-empty* sequences. The following principle is then consistent:

$$\begin{gathered}\forall F^{\tau} \forall G \forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n (F x_1 \dots x_n = G y_1 \dots y_n \rightarrow \\ F = G \wedge x_1 = y_1 \wedge \dots \wedge x_n = y_n), \text{ provided } \tau \text{ is recoverable}\end{gathered} \tag{38}$$

This principle says that propositions *do* have predicational structure so long as the entities involved are of recoverable types. It supports a naïve understanding (in terms of predicational structure) of one proposition being an instance of another if we are

[23] It was in effect proved by Russell [39, appendix B] and rediscovered by Myhill [37].

[24] Walsh [44] defends (35) by rejecting (18), and Kment [24] by rejecting the law of excluded middle (motivated by a ground-theoretic iterative conception of propositions).

[25] Informally: (37) implies that applying properties to a given proposition is a one-to-one operation for making propositions from properties of propositions. And the existence of such an operation is inconsistent for Cantorian reasons. Consider the property of being a proposition made in this way from a property that does not apply to the proposition so made. Does this property apply to the proposition made from it? It must if it doesn't, but it cannot if it does. See [22] for a formal derivation.

willing to say that the instance-of relation is well-defined only for generalizations about recoverable entities. This is the view I want to explore.

Since $\wedge$ and $\vee$ are of type $\langle\langle\rangle, \langle\rangle\rangle$, which is not recoverable, we must reject the naïve understanding (in terms of predicational structure) of one proposition being a conjunct or a disjunct of another. But this does not mean we must reject the idea of conjunctive and disjunctive structure. Instead, we can understand such structure in the same way we did in Section 3: conjunctions and disjunctions are formed in a sequence of stages from sets of propositions from earlier stages, with the initial stage now comprised of all propositions with a predicational structure. The resulting picture is a kind of logical atomism. There are atomic propositions, with the structure of predications involving recoverable entities, and logically complex propositions built from atomic propositions via conjunction and disjunction. Quantification over entities of recoverable types is recorded as predicational structure of the resulting generalizations. While being an instance of a universal generalization is structurally different from being a conjunct of a conjunction, the two relations play the same role in a theory of grounding, and likewise for existential generalizations and disjunctions.

The view responds to Krämer's puzzle by rejecting (7). Propositions only have quantificational structure in the case of recoverable types. Since $\langle\rangle$ is not recoverable, $\exists_{\langle\rangle}(\lambda p.p)$ lacks such structure. Only where there is such structure will true generalizations be grounded in their true instances. The view therefore accepts all and only the instances of (30) and (31) where $\tau_1, \ldots, \tau_n$ are all recoverable types. Models of this theory are given in Appendix F.

The view also avoids the problem of the grounds of truth ascriptions discussed in Section 5.1. In the models discussed in the appendix, the function that maps every proposition to its self-conjunction does correspond to a property of propositions, and is an available interpretation of a truth operator T.[26] So interpreted, the view prioritizes truth-grounding over instance-grounding in the same way that the popular Kripke-inspired views discussed in Section 4 do, although for very different reasons. Those theories hold that $T\exists pTp$ fails to ground $\exists pTp$ *despite* being an instance of it, for holistic reasons to do with the patterns of would-be grounding among all propositions. The present view, by contrast, holds that $T\exists pTp$ fails to ground $\exists pTp$ because it is *not* in the relevant sense an instance of it, despite the *sentence* "$T\exists pTp$" being an instance of "$\exists pTp$". There is no need to appeal to holistic considerations to determine patterns of logical grounding; we instead adopt a naïve theory of grounding relations as the result of chaining together true conjuncts, disjuncts, and instances.

[26] There are different options for how to interpret $\neg p$, such as (i) the involution of p (in which case $\neg\neg p = p$) or (ii) the involution of Tp (in which case $p \rightarrow (Tp \prec \neg\neg p)$). (Wilhelm [45] shows how similar decision points arise for grounding theorists quite generally.) Both are heterodox: option (i) because propositions don't ground their double-negations, and option (ii) because they don't *immediately* ground their double-negations (*pace* Fine [15] and Correia [7]) since the proposition that they are true is in between. I think the latter is arguably an attractive prediction, since "it's true that it's raining" seems like a good grounding explanation for "why is it not not raining?"; it also seems odd that $\neg\neg p$ should be only one level above p when TTp is two levels above it, as it were. That said, a view that validates the grounding orthodoxy for negation is described at the end of Appendix G, and Litland [35] argues that (i) might be motivated by a ground-theoretic account of propositional granularity.

7 Synthesis

Propositions have different structure from the sentences we use to express them. We conjoin sentences two at a time in a given order, but perhaps propositions can be conjoined many at a time and in no particular order. We generalize by combining predicates and quantifiers, but perhaps the propositions we thereby express have the structure of infinite conjunctions or disjunctions of their instances. While the nature and extent of these structural discrepancies is contested, the existence of some such discrepancies should not be. For propositions cannot have as much predicational structure as sentences (since (37) is inconsistent) nor as much conjunctive and quantificational structure (since (19) and (20) are jointly inconsistent).

Despite these limitative results, the space of consistent theories of propositional granularity is both vast and poorly understood. In choosing between such theories, we should not be dispassionate. Applications matter, and grounding is an illustrative case study. Theories about the grounds of conjunctions, disjunctions, and generalizations have significant implications about how fine-grained propositions, properties and relations must be. In this way theories of grounding can help guide our investigation into reality's granularity.

But while the pictures of reality's granularity that I have been exploring here were developed in the service of theories of grounding, they are independently interesting. The first illustrates the tradeoffs inherent in having as much conjunctive and instance structure as possible. The second illustrates the tradeoffs arising from having as much predicational structure as possible. Both pictures are novel and ascribe to reality a rich logical structure that departs in basic ways from the logical structure of our language.

There is an important respect in which my discussion of these two pictures has been misleading. As developed above, the pictures are competitors, since they disagree about the correct diagnosis of Krämer's puzzle. This was by design, both to keep separate ideas separate and to emphasize that two different type-theoretic diagnoses of that puzzle are possible, one that appeals to $(\lambda p.p)$ being monadic and another that appeals to it being of a non-recoverable type.

But if we step back from grounding puzzles and consider these pictures' distinctive claims about propositional granularity, they are not obviously in tension. The most distinctive feature of the view in Section 3 is that all true basic propositions are modally independent, while the most distinctive feature of the view in Section 6 is that basic propositions have internal predicational structure. I want to close by explaining how these two features can be combined to yield what is arguably a more attractive theory of reality's granularity than any of the views discussed so far. The formal details are given in Appendix G.

The picture that emerges can be seen as a blend of Russell's and Wittgenstein's logical atomisms. When it comes to properties of and relations among individuals, properties of and relations among those, and so on, propositions have Russellian predicational structure. But when it comes to properties of and relations had by propositions, properties of and relations among them, and so on, such predicational structure is inconsistent. Here things get Wittgensteinian. Generalizations are identified with conjunctions or disjunctions of their instances. All truths are ultimately grounded in logically independent atomic propositions (predications involving only *fundamental* entities) and the negations of such propositions. All standard grounding principles about truth, negation, conjunction, and disjunction can be validated, provided we treat Boolean connectives syncategorematically (as discussed in Section 5.3). But not all grounding relationships are logical, since propositions with predicational structure that predicate non-fundamental properties must be grounded in other ways. Looking beyond grounding, this theory offers a strong and concrete framework for any systematic investigation that appeals to logical structure in the world.[27]

Appendix A: Truth Condition Models

Here we give a precise characterization of the models described in Section 2. We there described two possible treatments of the connectives $\wedge$ and $\vee$ and two possible treatments of λ-terms; these are distinguished by subscripts below.

Let W be some non-empty set and $c \neq d$ be arbitrary urelements.

Definition 1 (Conjunction and disjunction)
$\bigwedge X = \{c\} \cup X$ if $X \neq \emptyset$ and W otherwise.
$\bigvee X = \{d\} \cup X$ if $X \neq \emptyset$ and $\emptyset$ otherwise.

Definition 2 (Propositions)
$P_0 = \mathcal{P}(W)$
$P_{n+1} = P_0 \cup \{\bigwedge X : X \subseteq P_n\} \cup \{\bigvee X : X \subseteq P_n\}$

Definition 3 (Ranks) $\kappa(\tau) = 0$ if τ is not monadic; $\kappa(\langle\tau\rangle) = 1 + \kappa(\tau)$.

Definition 4 (Domains)
D_e is some set
$D_{\langle\rangle} = \bigcup_{n\in\mathbb{N}} P_n$
$D_{\langle\tau\rangle} = P_{\kappa(\tau)}{}^{D_\tau}$
$D_{\langle\tau_1,\ldots,\tau_n\rangle} = D_{\langle\rangle}{}^{D_{\tau_1}\times\cdots\times D_{\tau_n}}$ for $n > 1$

Definition 5 (Truth conditions) $tc : D_{\langle\rangle} \to \mathcal{P}(W)$ such that
$tc(p) = p$ for $p \in \mathcal{P}(W)$
$tc(p) = \bigcap\{tc(x) : x \in p\}$ if $c \in p$
$tc(p) = \bigcup\{tc(x) : x \in p\}$ if $d \in p$

Definition 6 (Interpretation)
$[\![x]\!]^g = g(v)$ for variables v
$[\![Fa_1 \ldots a_n]\!]^g = [\![F]\!]^g([\![a_1]\!]^g, \ldots, [\![a_n]\!]^g)$
$[\![\neg]\!]^g(p) = W \backslash tc(p)$

[27] This paper has been an exercise in devil's advocacy. I reject all of the theories developed here, since I accept β-conversion, which I defend in [23].

$[\![\wedge_1]\!]^g(p, q) = \bigwedge\{p, q\}$
$[\![\vee_1]\!]^g(p), = \bigvee\{p, q\}$
$[\![\wedge_2]\!]^g(p, q) = p$ if $\{q, c\} \subseteq p$; q if $\{p, c\} \subseteq q$; and $\bigwedge\{p, q\}$ otherwise
$[\![\vee_2]\!]^g(p, q) = p$ if $\{q, d\} \subseteq p$; q if $\{p, d\} \subseteq q$; and $\bigvee\{p, q\}$ otherwise
$[\![\prec]\!]^g(p)(q) = \bigvee\{\bigwedge\{r_1, \ldots, r_n\} : p = r_1 \in \cdots \in r_n = q \neq p\}$
$[\![\forall_\tau]\!]^g(f) = \bigwedge\{f(x) : x \in D_\tau\}$
$[\![\exists_\tau]\!]^g(f) = \bigvee\{f(x) : x \in D_\tau\}$
$[\![(\lambda_1 x_1 \ldots x_n.\varphi)]\!]^g(y_1, \ldots, y_n) = tc([\![\varphi]\!]^{g[x_i \to y_i]})$
$[\![(\lambda_2 x_1^{\tau_1} \ldots x_n^{\tau_n}.\varphi)]\!]^g(y_1, \ldots, y_n) =$
(i)$[\![\varphi]\!]^{g[x_i \to y_i]}$ if $\exists f \in D_{\langle \tau_1, \ldots, \tau_n \rangle} \forall y_i \in D_{\tau_i} : f(y_1, \ldots, y_n) = [\![\varphi]\!]^{g[x_i \to y_i]}$,
(ii) $tc([\![\varphi]\!]^{g[x_i \to y_i]})$ otherwise.

Definition 7 (Validity) φ is *valid* := $tc([\![\varphi]\!]^g) = W$ for all g in any model

Appendix B: Plural Quantification and Full Ground

For every type τ we add a corresponding type τ^* for pluralities of entities of type τ. (Compare the 'extensional types' of [13].) Formally, we enrich our type system as follows: e is a type; for any type τ, τ^* is a type; for any types $\tau_1, \ldots, \tau_n$, $\langle \tau_1, \ldots, \tau_n \rangle$ is a type; nothing else is a type. Domains for non-plural types are defined as before. $D_{\tau^*} = \mathcal{P}(D_\tau)$. For every type τ, we enrich our language with an 'is one of' connective $\in_\tau$ of type $\langle \tau, \tau^* \rangle$ such that $[\![\in_\tau]\!]^g(x, Y) = W$ if $x \in Y$ and $= \emptyset$ otherwise.

Now to full grounding. Intuitively, a way for Γ to fully ground p is given by a collection of propositions that can be organized in such a way that any such proposition not in Γ is assigned some such propositions that immediately fully ground it (by being either all of its conjuncts or some of its disjuncts), every maximal chain of these immediate full grounding relationships begins in Γ and ends at p, and every member of Γ is used in the process.

Definition 8 *X is a way for Γ to fully ground p* if and only if, for some non-trivial rooted tree $G = \langle V, E, v \rangle$ and surjective function $f : V \to X$,

1. $f(v) = p$
2. $\{f(x) : x$ has no children$\} = \Gamma$
3. if x is a child of y, then $f(x)$ is either a conjunct or a disjunct or $f(y)$
4. if $f(x) \notin \Gamma$ and q is a conjunct of $f(x)$, then $q = f(y)$ for some child y of x.

Finally, we enrich our language with a full grounding connective $<$ of type $\langle \langle \rangle^*, \langle \rangle \rangle$, subject to the interpretation:

$$[\![<]\!]^g(\Gamma, p) = \bigvee \left\{ \bigwedge X : X \text{ is a way for } \Gamma \text{ to fully ground} p \right\}.$$

(Parallel strategies for introducing a full grounding connective in the context of the other models of grounding discussed below are straightforward.)

This clause validates all principles of the pure logic of (strict) ground in [16] (isolated in [11]), as well as the 'internality' of full grounding: $(\Gamma < p) \to \Box(\forall q(q \in_{\langle \rangle}$

$\Gamma \to q) \to (\Gamma < p))$. Partial grounds are all and only the parts of full grounds: $(p \prec q) \leftrightarrow \exists\Gamma(p \in_{\langle\rangle} \Gamma \wedge \Gamma < q)$. But not every full ground together with a partial ground is a full ground: $p \vee (q \wedge r)$ may be fully grounded in p alone and partially grounded in q without being fully grounded in p together with q.

The clause also validates the following controversial but popular principle about iterated grounding: $(\Gamma < p) \to (\Gamma < (\Gamma < p))$.[28] The main reservation about that principle in the literature is that $\Gamma < p$ and $\Gamma < q$ can have different full grounds when p and q are different propositions. But that is validated too: $((\Gamma < p) \wedge (\Gamma < q)) \to (p \neq q \to \neg\forall\Delta((\Delta < (\Gamma < p)) \leftrightarrow (\Delta < (\Gamma < q))))$. There is no incompatibility here because $<$ is a many-one relation: true grounding claims typically have many full grounds.

Appendix C: Atomist Models

Here we give a precise characterization of the modified model construction described in Section 3. This involves (i) changing the definition of propositional domains, (ii) introducing a normalization operation that replaces the truth condition operation in the clause for λ-abstraction, and (iii) introducing parallel changes in the clause for negation.

P_0 is now a set of basic *atomic propositions*, imbued with a negation operation $\nu : P_0 \to P_0$ such that $\nu(p) \neq \nu(\nu(p)) = p$ for all $p \in P_0$. Domains are defined as in Appendix A, with two changes:

$$\begin{aligned} P_{n+1} &= P_0 \cup \{\{c\} \cup X : X \subseteq P_n\} \cup \{\{d\} \cup X : X \subseteq P_n\} \\ D_{\langle\tau\rangle} &= {P_{\kappa(\tau)+2}}^{D_\tau} \end{aligned}$$

The first change is to allow for a trivial tautology $\{c\}$ and contradiction $\{d\}$. We make a parallel adjustment in the interpretation of $\prec$:

$$[\![\prec]\!]^g(p, q) = \{d\} \cup \{\{c, r_1, \ldots, r_n\} : p = r_1 \in \cdots \in r_n = q \neq p\}$$

We now define an operation for turning level $n + 3$ propositions into Boolean-equivalent level $n + 2$ propositions. The intuitive idea is that every conjunction is replaced with its conjunctive normal form and every disjunction with its disjunctive normal form. For example, suppose we have a level-3 conjunction. We first turn it into a level-3 conjunction of disjunctions, by replacing every level-0 conjunct with its singleton disjunction and replacing every conjunctive conjunct with the singleton disjunctions of its members. Next, we do the same to this conjunctions' disjuncts, yielding a level-3 conjunction of disjunctions of conjunctions. Then, we invert each of its conjuncts from a disjunction of conjunctions to a conjunction of disjunctions, yielding a level-3 conjunction of conjunctions of disjunctions. We then merge the conjuncts together, yielding a level-2 conjunction of disjunctions equivalent to our original proposition; for an example see footnote (12). By iterating this procedure n

[28] This principle is endorsed by [4, 10, 32]; see [33] for a review of the literature on iterated grounding.

times any proposition of level $n+2$ can be reduced to a level-2 normal form. More precisely:

Definition 9 (Sign) $\sigma(p) = c$ if $c \in p$ and $\sigma(p) = d$ if $d \in p$.

Definition 10 (Members) $m(p) = p \backslash \{\sigma(p)\}$

Definition 11 (Involution) Let * be the operation on $D_{\langle\rangle} \cup \{c, d\}$ such that
$p^* = \nu(p)$ for $p \in P_0$
$c^* = d$
$d^* = c$
$p^* = \{x^* : x \in p\}$ for $p \in D_{\langle\rangle} \backslash P_0$

Definition 12 (Polarization)
$p^\dagger = \{\sigma(p)\} \cup \{q : q \in p$ and $\sigma(q) = \sigma(p)^*\} \cup \{\{\sigma(p)^*, q\} : q \in p \cap P_0\} \cup \{\{\sigma(p)^*, q\} : \exists r \in p$ s.t. $q \in m(r)$ and $\sigma(r) = \sigma(p)\}$

Definition 13 (Inversion) If $m(p) = \emptyset$, then $\iota(p) = \{\sigma(p)^*, p\}$. Otherwise, $\iota(p) = \{\sigma(p)^*\} \cup \{\{\sigma(p)\} \cup \text{image}(f) : f$ a choice function on $m(p)\}$.

Definition 14 (Flattening) $f(p) = \bigcup\{\iota(q^\dagger) : q \in m(p^\dagger)\}$

Definition 15 (Levels) $\mathfrak{l}(p) = \min\{n : p \in P_n\}$

Definition 16 (Normal form) Let $norm : D_{\langle\rangle} \to P_2$ s.t. $norm(p) = f^{\mathfrak{l}(p)-2}(p)$ if $\mathfrak{l}(p) > 2$, and $norm(p) = p$ otherwise.

We interpret our language as in Appendix A with three exceptions: (i) we modify the clause for $\prec$ as described above, (ii) we replace tc with $norm$ in the clause for λ-abstraction, and (iii) we modify the clauses for $\neg$ as follows:

$$[\![\neg]\!]^g(p) = norm(p^*)$$

Finally, we define a notion of truth conditions as follows:

Definition 17 (Worlds) $W = \{w \subseteq P_0 : \forall p \in P_0, p \in w \leftrightarrow \nu(p) \notin w\}$

Definition 18 (Truth conditions) $tc : D_{\langle\rangle} \cup \{c, d\} \to \mathcal{P}(W)$ such that
$tc(p) = \{w : p \in w\}$ for $p \in P_0$
$tc(c) = W$
$tc(d) = \emptyset$
$tc(p) = \bigcap\{tc(x) : x \in p\}$ if $c \in p$
$tc(p) = \bigcup\{tc(x) : x \in p\}$ if $d \in p$

As before, a formula is valid if and only if it has trivial truth conditions in every model relative to every assignment.

Appendix D: Syncategoremtic Models

This appendix explains how to modify the above construction to model the theory sketched in Section 5.3, which endorses (30) and (31) (formulated in terms of new quantifiers which take relations as arguments), and in which conjunction and disjunction are formalized not as connectives but as syntactic operations for forming new formulas from finite sets of formulas.

We first modify the definition of higher-order domains to give a uniform treatment of monadic and polyadic types. Ranks and domains are now defined as follows: $\kappa(e) = \kappa(\langle\rangle) = 0$; $\kappa(\langle\tau_1, \ldots, \tau_n\rangle) = \max(\kappa(\tau_1), \ldots, \kappa(\tau_n)) + 1$; $D_{\langle\tau_1,\ldots,\tau_n\rangle} = P_{\kappa(\langle\tau_1,\ldots,\tau_n\rangle)+1}{}^{D_{\tau_1}\times\cdots\times D_{\tau_n}}$ (for $n > 0$).

We then interpret set conjunction and disjunction in the obvious way: $[\![\bigwedge\Gamma]\!]^g = \{c\} \cup \{[\![\varphi]\!]^g : \varphi \in \Gamma\}$, and likewise for disjunction. Similarly for universal and existential quantifiers applying to polyadic predicates: $[\![\forall_{\tau_1,\ldots,\tau_n}]\!]^g(f) = \{c\} \cup \{f(x_1, \ldots, x_n) : x_i \in D_{\tau_i}\}$, and likewise for existential quantifiers.

The rest of the construction is as before with the obvious adjustments.[29]

Appendix E: Schematic Instance Structure

In this appendix we establish the consistency of the schemas:

$$\forall p(inst(p, \exists x\varphi) \leftrightarrow \exists x(p = \varphi))$$
$$\forall p(inst(p, \forall x\varphi) \leftrightarrow \exists x(p = \varphi))$$

For convenience, we will operate in a higher-order language of the sort discussed in Section 1, where quantifier prefixes are treated as syncategorematic variable-binding sentential operators. That is, $\exists x$ is a single expression that combines with a formula to yield a formula; however, unlike other expressions with this syntactic behavior (i.e., of type $\langle\langle\rangle\rangle$), we do not assign a semantic value to $\exists x$, but instead assign semantic values directly to formulas $\exists x\varphi$, as is standard in Tarskian model-theory for first-order languages.

The model construction has some high-level similarities to the ones discussed in Appendices A and C. In both cases, since we can recover the instances of a generalization from that generalization, Cantor's theorem implies that not all sets of propositions can be all and only the instances of some generalization. The main difference from the previous constructions is that generalizations can be instances of themselves. Another difference is a reversal in what is possible regarding conjunction and negation: the present construction is compatible with distinct propositions always having distinct negations (although for simplicity we will start off with models where this fails), but not with distinct pairs of propositions always having

[29] Having $\bigwedge$ and $\bigvee$ apply to arbitrary sets of formulas would require two additional modifications: (i) adding transfinite stages in constructing propositional domains, and (ii) generalizing the flattening and normalization operations to apply to propositions of transfinite level. (The latter can be done by letting the flattening of a conjunction/disjunction whose level is a limit ordinal be the conjunction/disjunction of the normalizations of its conjuncts/disjuncts.)

distinct conjunctions (given the inconsistency of instance structure and conjunctive structure).

I'll begin with an informal discription of the models. There are two distinguished 'boring' propositions, one true and one false. Each has itself and only itself as an instance. Every categorematic predication (i.e., every formula that is neither a free variable nor a generalization) denotes one of these two propositions. Generalizations are individuated by their instances. So the boring truth is $\forall p(p \vee \neg p)$ (equivalently, $\exists p(p \vee \neg p)$), since its instances are true categorematic predications (i.e., the boring truth), and the only proposition with the boring truth as its only instance is the boring truth. Likewise, the boring falsehood is $\forall p(p \wedge \neg p)$ (equivalently, $\exists p(p \wedge \neg p)$).

Now consider $\exists p \neg p$ and $\forall p \neg p$. These two propositions have the same instances: namely, both boring propositions. So they cannot be either of the two boring propositions. Now, if we didn't have sentences like $\exists q \exists p \neg p$ and $\exists pp$, we could get by with just these four propositions. This is because, for any formula φ and assignment g, it will turn out that $[\![\varphi]\!]^g$ will be one of these four propositions as long as any occurrences of vacuous propositional quantifiers or of propositional variables in φ are in the scope of some categorematic operator. But the possibility of vacuous propositional quantification and of propositional variables occurring only under quantifiers complicates things.

First, notice that, for any proposition p, $\forall xp$ has p as its only instance. Since, as we have seen, some propositions have more than one instance, it follows that there must be infinitely many propositions – and not only infinitely many propositions with a single instance. For example, $\forall p \forall x \forall q(p \rightarrow q)$ has not only the boring truth and falsehood as instances, but also the proposition whose only instance is the proposition that has the boring truth and boring falsehood as its only instances. $\forall p \forall x \forall q \forall y \forall r(p \rightarrow (q \rightarrow r))$ has more instances still, etc.

Next, consider bound propositional variables occurring only under quantifiers. For example, both $\exists pp$ and $\forall pp$ have every proposition as an instance. In fact, there is an infinite sequence of propositions with infinitely many instances: the instances of $\exists p \forall xp$ are all propositions that have a single instance; the instances of $\forall p \exists x \exists yp$ are all the propositions that have a single instance and whose instance has a single instance; etc. Fortunately, it can be show by induction on the complexity of formulas that these are the only propositions with infinitely many instances expressed by any closed sentence. It is this fact that allows us to build models of instance structure.

We now turn to a formal description of the models. Let $D_{\langle\rangle} = \mathbb{Z} \backslash \{0\}$. Let u be a partial function from $\mathcal{P}(D_{\langle\rangle}) \rightarrow D_{\langle\rangle}$ satisfying the following conditions:

- u is injective
- u is defined on all finite $X \subseteq D_{\langle\rangle}$
- u is defined on $\{f^n(p) : p \in D_{\langle\rangle}\}$, for all $n \in \mathbb{N}$, where $f(p) := u(\{p\})$
- $-u(X) = u(\{-p : p \in X\})$ if u is defined on X
- $u(\{1\}) = 1$
- $u(X) > 0$ if and only if $p > 0$ for all $p \in X$

While it is possible (but tedious) to explicitly specify such a function, the existence of such a function is easily seen by cardinality considerations. Intuitively, $u(X)$ is the universal generalization whose instances are all and only the members of X, and $-u(\{-p : p \in X\})$ is the corresponding existential generalization.

We let D_e be an arbitrary set and $D_{\langle\tau_1\ldots\tau_n\rangle} = \{1, -1\}^{D_{\tau_1}\times\cdots\times D_{\tau_n}}$. We then interpret our language as follows:

$$\llbracket x \rrbracket^g = g(x)$$
$$\llbracket Fa_1 \ldots a_n \rrbracket^g = \llbracket F \rrbracket^g(\llbracket a_1 \rrbracket^g, \ldots, \llbracket a_n \rrbracket^g)$$
$$\llbracket \neg \rrbracket^g(p) = 1 \text{ iff } p < 0$$
$$\llbracket \wedge \rrbracket^g(p, q) = 1 \text{ iff } p > 0 \text{ and } q > 0$$
$$\llbracket \forall x\varphi \rrbracket^g = u\{\llbracket \varphi \rrbracket^{g'} : g' \text{ an } x\text{-variant of } g\}$$
$$\llbracket \exists x\varphi \rrbracket^g = -u\{-\llbracket \varphi \rrbracket^{g'} : g' \text{ an } x\text{-variant of } g\}$$
$$\llbracket inst \rrbracket^g(p, q) = 1 \text{ iff } \exists X((p \in X \wedge q = u(X)) \vee (-p \in X \wedge q = -u(X)))$$

It can be shown by induction on the complexity of formulas that the clauses for quantifiers are well-defined given the conditions on u.

I'll now describe two natural ways in which the construction might be modified or extended. First, in order to validate double-negation equivalence (and hence the injectivity of negation), we could replace the above interpretation of negation with the syncategorematic clause:

$$\llbracket \neg\varphi \rrbracket^g = -\llbracket \varphi \rrbracket^g$$

Second, we can prune down the model in a natural way to eliminate arbitrary structure and superfluous propositions like $u(\emptyset)$. To do this, start with a model of the kind just described, and let the propositional domain $D^*_{\langle\rangle}$ of the new model be all and only the propositions denoted by closed sentences in the old model. $D^*_{\langle\rangle}$ is clearly a subset of the image of u. Moreover, it can be shown that, for any $p, q \in D^*_{\langle\rangle}$, if $u^{-1}(p) \cap D^*_{\langle\rangle} = u^{-1}(q) \cap D^*_{\langle\rangle}$, then $u^{-1}(p) = u^{-1}(q)$. So we can generate a new model by replacing u with the function $u^* : u^{-1}(p) \cap D^*_{\langle\rangle} \mapsto p$. Models generated in this way are unique up to isomorphism.

Appendix F: Predicational Models

The model construction here repurposes many of the ideas from Appendices A and C. We first characterize the recoverable types, entities of which are constituents of atomic propositions, as follows:

Definition 19 (Recoverable types) Let R be the smallest set of types containing e that is closed under the formation of *non-empty* sequences.

Next, we define our domains. Propositional domains are constructed as in Appendix C, except with the level-0 propositions P_0 now treated as having additional structure, *viz* predications involving recoverable entities. Recoverable properties and relations are identified with functions from entities of the relevant types to truth conditions. Non-recoverable properties and relations are identified with arbitrary functions from entities of the relevant types to propositions. The distinction here between recoverable and non-recoverable types is similar to that between monadic and polyadic types in Appendix A.

Definition 20 (Domains) For some non-empty W and urelements $c \neq d$:
$D_e =$ some set
$D_{\langle \tau_1,\ldots,\tau_n \rangle} = \mathcal{P}(W)^{D_{\tau_1} \times \cdots \times D_{\tau_n}}$ for $\langle \tau_1, \ldots, \tau_n \rangle \in R$
$D_{\langle \tau_1,\ldots,\tau_n \rangle} = D_{\langle\rangle}^{D_{\tau_1} \times \cdots \times D_{\tau_n}}$ for $\langle \tau_1, \ldots, \tau_n \rangle \notin R$
$D_{\langle\rangle} = \bigcup_{n \in \mathbb{N}} P_n$
$P_0 = \{\langle f, \langle x_1, \ldots, x_n \rangle\rangle : f \in D_{\langle \tau_1,\ldots,\tau_n \rangle}, x_i \in D_{\tau_i}, \langle \tau_1, \ldots, \tau_n \rangle \in R\}$
$P_{n+1} = P_0 \cup \{\{c\} \cup X : X \subseteq P_n\} \cup \{\{d\} \cup X : X \subseteq P_n\}$ (same as in Appendix C)

Definition 21 (Truth conditions) $tc : \langle f, \langle x_1, \ldots, x_n \rangle\rangle \in P_0 \mapsto f(x_1, \ldots, x_n)$; tc is defined as in Appendix C for $p \in D_{\langle\rangle} \backslash P_0$.

Definition 22 (Property negation) $' : \bigcup_{\tau \in R \backslash \{e\}} D_\tau \to \bigcup_{\tau \in R \backslash \{e\}} D_\tau$ such that $f'(x_1, \ldots, x_n) = W \backslash f(x_1, \ldots, x_n)$ for all $\langle x_1, \ldots, x_n \rangle \in \text{domain}(f)$.

Definition 23 (Involution) $^* : \langle f, \langle x_1, \ldots, x_n \rangle\rangle \in P_0 \mapsto \langle f', \langle x_1, \ldots, x_n \rangle\rangle$; * is defined as in Appendix C on $D_{\langle\rangle} \backslash P_0$.

Since instance structure is no longer a species of conjunctive/disjunctive structure, we need to add it to the list of kinds of immediate non-factive grounding relations in terms of which we interpret $\prec$, which we do as follows.

Definition 24 (Projection functions) $\pi_i(\langle x_1, \ldots, x_n \rangle) = x_i$

Definition 25 (Immediate non-factive grounds) For $p, q \in D_{\langle\rangle}$, $p \propto q :=$ either $p \in q$, or $\pi_1(q) \in \bigcup_{\tau_i \in R}\{[\![\forall_{\tau_1,\ldots,\tau_n}]\!], [\![\exists_{\tau_1,\ldots,\tau_n}]\!]\}$ (see below) and $\pi_2(q) = \langle \pi_1(p) \rangle$.

Definition 26 (Surrogates for truth conditions) $s : \mathcal{P}(W) \to P_0$ s.t. $s(X) = \langle \{X\}^{D_{\langle e \rangle}}, \langle \{W\}^{D_e} \rangle\rangle$ (gloss: self-identity is such that an X-world obtains)

Definition 27 (Interpretation) We interpret $\wedge$ and $\vee$ as in Appendix A.
$[\![F^\tau a_1 \ldots a_n]\!]^g = \langle [\![F]\!]^g, \langle [\![a_1]\!]^g, \ldots, [\![a_n]\!]^g \rangle\rangle$ if $\tau \in R$ and $[\![F]\!]^g([\![a_1]\!]^g, \ldots, [\![a_n]\!]^g)$ if $\tau \notin R$
$[\![T]\!]^g(p) = \{c, p\}$ (alternatively $\{d, p\}$)
$[\![\neg]\!]^g(p) = p^*$ (alternatively $[\![T]\!]^g(p)^*$; see note 26)
$[\![\prec]\!]^g(p, q) = \{d\} \cup \{\{c, r_1, \ldots, r_n\} : p = r_1 \propto \cdots \propto r_n = q \neq p\}$
$[\![\forall_\tau]\!]^g(f) = \bigcap\{f(x) : x \in D_\tau\}$ for $\tau \in R$ and $s(\bigcap\{tc(f(x)) : x \in D_\tau\})$ for $\tau \notin R$; similarly for $\forall_{\tau_1,\ldots,\tau_n}$ when $n > 1$
$[\![\exists_\tau]\!]^g(f) = \bigcup\{f(x) : x \in D_\tau\}$ for $\tau \in R$ and $s(\bigcup\{tc(f(x)) : x \in D_\tau\})$ for $\tau \notin R$; similarly for $\exists_{\tau_1,\ldots,\tau_n}$ when $n > 1$
$[\![(\lambda x_1^{\tau_1} \ldots x_n^{\tau_n}.\varphi)]\!]^g(y_1, \ldots, y_n) = tc([\![\varphi]\!]^{g[x_i \to y_i]})$ for $\langle \tau_1, \ldots, \tau_n \rangle \in R$ and $[\![\varphi]\!]^{g[x_i \to y_i]}$ for $\langle \tau_1, \ldots, \tau_n \rangle \notin R$

Appendix G: Atomism with Predicational Structure

This appendix shows how the ideas of Appendices C and F can be combined to model a version of logical atomism where atomic propositions decompose into predications of fundamental properties and relations.

We model fundamental entities (logical atoms) as members of a set A of variables of our language such that, if $a_0^{\langle\tau_1,\ldots,\tau_n\rangle} \in A$, then (i) $\langle\tau_1,\ldots,\tau_n\rangle \in R$ and (ii) for all τ_i, there is some $a_i^{\tau_i} \in A$. Atomic propositions correspond to predications $\ulcorner a_0 a_1 \ldots a_n \urcorner$ involving these variables (although, for reasons that will emerge, they are not themselves members of $D_{\langle\rangle}$). We will now define a set F the members of which correspond to Boolean combinations of logical atoms:

Definition 28 (Fundamentally based propositions)
$F_0 = \{\ulcorner a_0^{\langle\tau_1,\ldots,\tau_n\rangle} a_1^{\tau_1} \ldots a_n^{\tau_n} \urcorner : a_i \in A\}$
$F_{m+1} = F_0 \cup \{\{n, p\} : p \in F_m\} \cup \{\{c\} \cup X : X \subseteq F_m\} \cup \{\{d\} \cup X : X \subseteq F_m\}$
$F = \bigcup_n F_n$

Note the new structure building operation: $\{n, p\}$ is the negation of p.

Definition 29 (Domains) Drawing on Appendices C and F:
$D_e = \{a^e : a \in A\}$
$D_{\langle\tau_1,\ldots,\tau_n\rangle} = F^{D_{\tau_1}\times\cdots\times D_{\tau_n}}$ for $\tau_i \in R$
$P_0 = \{\langle f, \langle x_1,\ldots,x_n\rangle\rangle : f \in D_{\langle\tau_1,\ldots,\tau_n\rangle}, x_i \in D_{\tau_i}, f(x_1,\ldots,x_n) \in F_0\}$
$P_{m+1} = \{\{n, p\} : p \in P_m\} \cup \{\{c\} \cup X : X \subseteq P_m\} \cup \{\{d\} \cup X : X \subseteq P_m\} \cup$
$\{\langle f, \langle x_1,\ldots,x_n\rangle\rangle : f \in D_{\langle\tau_1,\ldots,\tau_n\rangle}, x_i \in D_{\tau_i}, f(x_1,\ldots,x_n) \in F_{m+1}\}$
$D_{\langle\rangle} = \bigcup_{n\in\mathbb{N}} P_n$
$D_{\langle\tau_1,\ldots,\tau_n\rangle} = D_{\langle\rangle}^{D_{\tau_1}\times\cdots\times D_{\tau_n}}$ if any $\tau_i \notin R$
$D_{\langle\tau\rangle} = P_{\kappa(\tau)+3}{}^{D_\tau}$ for $\tau \notin R$

We now define three functions: $\mathfrak{a}$ maps every member of A to the corresponding entity in D; $\cdot^{\uparrow}$ maps every member of $D_{\langle\rangle}$ to the member of F that results from "unpacking" the content of its predicational constituents; $\cdot^{\downarrow}$ maps every member of F to the corresponding member of $D_{\langle\rangle}$.

Definition 30 (Logical atoms) $\mathfrak{a} : a^\tau \to D_\tau$ s.t.
$\mathfrak{a}(a^e) = a$
$\mathfrak{a}(a_0^{\langle\tau_1,\ldots,\tau_n\rangle})(\mathfrak{a}(a_1^{\tau_1}),\ldots,\mathfrak{a}(a_n^{\tau_1})) = \ulcorner a_0 a_1 \ldots a_n \urcorner$
$\mathfrak{a}(a_0^{\langle\tau_1,\ldots,\tau_n\rangle})(x_1,\ldots,x_n) = \{d\}$ if any $x_i \notin \text{image}(\mathfrak{a})$

Remark 31 The third clause in the definition of $\mathfrak{a}$ says that applying an atomic relation to a non-atomic entity yields the trivial contradiction. This is somewhat arbitrary, but there appears to be no more principled option. Note that this possibility does not arise if A contains no higher-order predicates. This is a count in favor of the view that all atomic/fundamental properties and relations apply to individuals, held by Lewis

[30] against Armstrong [1] (who held that laws of nature are a matter of a fundamental relation of nomic necessitation holding between properties of individuals). Note also that no denotations of logical connectives will be in the image of $\mathfrak{a}$ – logicality is distinguished from fundamentality (as Dorr [12] recommends against Sider [41]). This is a natural separation once we grant that only entities of recoverable types are fundamental, and also when conjunctive/disjunctive structure is modeled as an operation on sets of propositions rather than on pairs of propositions.

Definition 32 (Unpacking) Let $\cdot^{\uparrow} : D_{\langle\rangle} \to F$ s.t.
(i) $p^{\uparrow} = \pi_1(p)(\pi_2(p))$ if p is an ordered pair, and
(ii) $(\{*\} \cup X)^{\uparrow} = \{*\} \cup \{p^{\uparrow} : p \in X\}$ for $* \in \{n, c, d\}$.

Definition 33 (Correspondence) Let $\cdot^{\downarrow} : F \to D_{\langle\rangle}$ s.t.
(i) $\ulcorner a_0 a_1 \ldots a_n \urcorner^{\downarrow} = \langle \mathfrak{a}(a_0), \langle \mathfrak{a}(a_1), \ldots, \mathfrak{a}(a_n)\rangle\rangle$ for $\ulcorner a_0 a_1 \ldots a_n \urcorner \in F_0$, and
(ii) $(\{*\} \cup X)^{\downarrow} = \{*\} \cup \{p^{\downarrow} : p \in X\}$ for $* \in \{n, c, d\}$.

Note that the image of $\cdot^{\uparrow}$ is F and that $p^{\uparrow\downarrow\uparrow} = p^{\uparrow}$ for all $p \in D_{\langle\rangle}$.

As in Appendix C, the interpretations of $\neg$ and of λ-abstraction appeal to an operation $norm$ mapping every member of $D_{\langle\rangle}$ to a Boolean-equivalent normal form of degree ≤ 3 (not 2, since now we have negation structure to account for). There are many natural such operations, so we omit an explicit characterization. The clauses for negation and λ-abstraction are then:

$$\llbracket \neg \rrbracket^g(p) = norm(\{n, p^{\uparrow\downarrow}\})$$
$$\llbracket (\lambda x_1^{\tau_1} \ldots x_n^{\tau_n}.\varphi) \rrbracket^g(y_1, \ldots, y_n) = (\llbracket \varphi \rrbracket^{g[x_i \mapsto y_i]})^{\uparrow} \text{ for } \tau_i \in R$$
$$\llbracket (\lambda x_1^{\tau_1} \ldots x_n^{\tau_n}.\varphi) \rrbracket^g(y_1, \ldots, y_n) = \llbracket \varphi \rrbracket^{g[x_i \mapsto y_i]} \text{ if some } \tau_i \notin R, n > 1$$
$$\llbracket (\lambda x^{\tau}.\varphi) \rrbracket^g(y) = norm((\llbracket \varphi \rrbracket^{g[x_i \mapsto y_i]})^{\uparrow\downarrow}) \text{ for } \tau \notin R$$

We now turn to the interpretation of quantifiers. As in Appendices A and C, if f is a monadic property of a non-recoverable type, then its universal generalization is the conjunction of its instances. (There is no primitive notion of universal generalization applying to non-recoverable polyadic relations.) The situation for recoverable types is more complicated. The denotation of a recoverable-type universal quantifier $\forall_{\tau_1,\ldots,\tau_n}$ cannot always map $f \in D_{\langle\tau_1,\ldots,\tau_n\rangle}$ to the conjunction of the instances of the resulting generalization, since the image of f may be unbounded in level and so have no conjunction; nor can it map f to appropriate truth conditions, as in Appendix F, since worlds are absent from the present model construction. Instead, we interpret the universal quantifier as the function mapping f to the conjunction of normalizations of propositions its image (where $norm$ is defined on F as in $D_{\langle\rangle}$). This is inelegant but harmless, since application at recoverable types corresponds to ordered-pair formation rather than to function application, so the clauses below won't threaten the claim that the only immediate grounds of a generalization are it instances.

$$\llbracket \forall_{\tau} \rrbracket^g(f) = \{c\} \cup \{f(x) : x \in D_{\tau}\} \text{ for } \tau \notin R$$
$$\llbracket \exists_{\tau} \rrbracket^g(f) = \{d\} \cup \{f(x) : x \in D_{\tau}\} \text{ for } \tau \notin R$$
$$\llbracket \forall_{\tau_1,\ldots,\tau_n} \rrbracket^g(f) = \{c\} \cup \{norm(f(x_1, \ldots, x_n)) : x_i \in D_{\tau_i}\} \text{ for } \tau_i \in R$$
$$\llbracket \exists_{\tau_1,\ldots,\tau_n} \rrbracket^g(f) = \{d\} \cup \{norm(f(x_1, \ldots, x_n)) : x_i \in D_{\tau_i}\} \text{ for } \tau_i \in R$$

We now define immediate non-factive grounds. There are three new cases to consider: negated conjunctions and disjunctions, negated negations, and predications of non-fundamental recoverable properties and relations; unlike all of the previous constructions, this third case is an example of grounding not attributable to Boolean or quantificational structure.

Definition 34 (Immediate non-factive grounds) $p \propto q := p, q \in D_{\langle\rangle}$ and one of the following conditions holds:

1. $\pi_1(q) \in \bigcup_{\tau_i \in R}\{[\![\forall_{\tau_1,\ldots,\tau_n}]\!], [\![\exists_{\tau_1,\ldots,\tau_n}]\!]\}$ and $\pi_2(q) = \langle\pi_1(p)\rangle$
2. $\pi_1(q) \notin \bigcup_{\tau_i \in R}\{[\![\forall_{\tau_1,\ldots,\tau_n}]\!], [\![\exists_{\tau_1,\ldots,\tau_n}]\!]\}$, $\pi_1(q)(\pi_2(q))^{\downarrow} = p$, and $p \neq q$
3. $n \notin q$ and $p \in q$
4. $q = \{n, r\}, n \notin r, s \in r$ and $p = \{n, s\}$, for some r, s
5. $q = \{n, \{n, p\}\}$

We interpret $\prec$ using $\propto$ as in Appendix F. To model full grounding, we modify the definition of ways of fully grounding a proposition as below, and then give the same interpretation of $<$ as in Appendix B.

Definition 35 *X is a way for Γ to fully ground p* if and only if, for some non-trivial rooted tree $G = \langle V, E, v\rangle$ and surjective function $f : V \to X$,

1. $f(v) = p$
2. $\{f(x) : x$ has no children$\} = \Gamma$
3. if x is a child of y, then $f(x) \propto f(y)$
4. if $q \in X\backslash\Gamma$ and either (i) $c \in q$, (ii) for some r, $q = \{n, r\}$ and $d \in r$, or (iii) $\pi_1(q) \in \bigcup_{\tau_i \in R}\{[\![\forall_{\tau_1,\ldots,\tau_n}]\!]\}$, then $\{r : r \propto q\} \subseteq X$.

We interpret predication as in Appendix F: as ordered-pair formation at recoverable types and as function application at non-recoverable types. While both options for $\wedge$ and $\vee$ described in Appendix A remain available, the second seems even more natural here, given the precedent of $\neg$ failing to express the model-theoretic negation operation.

One attraction of this theory is that the notions of being a *fundamental* entity (i.e., in the image of $\mathfrak{a}$) and of being an *atomic proposition* (i.e., being a predication involving only such entities) are definable in the object language:

$$\text{Atomic} := (\lambda p.(\lambda q.q \prec p) = (\lambda q.q \prec q))$$
$$\text{Fundamental}_e := (\lambda x^e.x = x)$$
$$\text{Fundamental}_\tau := (\lambda x^\tau.x \neq x) \text{ for } \tau \notin R$$
$$\text{Fundamental}_{\langle\tau_1,\ldots,\tau_n\rangle} := (\lambda F.\exists x_1^{\tau_1}\ldots\exists x_n^{\tau_n}(\text{Atomic}(Fx_1\ldots x_n))) \text{ for } \tau_i \in R$$

Note, finally, that the constructions here can be modified in the manner of Appendix D to validate the kind of view discussed in Section 5.3. On this view, all generalizations – even of polyadic relations at non-recoverable types – carry a record of their instances, which will be all and only their immediate grounds. At recoverable types, they do so by having predicational structure which records the kind of generalization they are and what property or relation is being generalized. Such structure is inconsistent at non-recoverable types: there, generalizations are identified with conjunctions/disjunctions of their instances. On this view conjunction and disjunction are not relations between propositions but operations for forming new propositions from sets of propositions. It is then natural to regiment negation in a similarly syncategorematic way. We can then have negated formulas express the model-theoretic negation of the proposition expressed by their unnegated counterparts, without restricting classical quantification theory, as discussed in Section 5.3. Doing so validates the standard principles about the grounds of negated propositions from Fine [15].

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