



Agglomerative Algebras

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Received: 21 February 2017 / Accepted: 2 October 2018 / Published online: 8 November 2018
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Abstract

This paper investigates a generalization of Boolean algebras which I call *agglomerative algebras*. It also outlines two conceptions of propositions according to which they form an agglomerative algebra but not a Boolean algebra with respect to conjunction and negation.

Keywords Fineness of grain · Aboutness · Subject matter · Propositions

Many philosophers believe that propositions form a Boolean algebra under the operations of conjunction and negation. The most familiar version of this view holds that necessarily materially equivalent propositions are identical. The present investigation is motivated by the idea that Boolean algebras are only half of the story. Propositions differ along two axes: a *logical* dimension and a *non-logical* dimension. Propositions' logical contents do form a Boolean algebra with respect to conjunction and negation. But propositions with the same logical content can differ in their non-logical content. A proposition's non-logical content is the same as its negation's, the non-logical content of a conjunction is the aggregate of the non-logical contents of its conjuncts, and there is a proposition \top with trivial logical content and trivial non-logical content. One version of this idea, in the spirit of recent work by Yablo [12] and and Yalcin [13, 14], identifies the non-logical content of a proposition with its subject matter. A different version of the idea, which I explore in other work, identifies the non-logical content of a proposition with the plurality of all entities that the proposition is about. We will return to these views about propositional fineness of grain in Section 7. The bulk of this paper, however, abstracts away from such potential applications. Its aim is to provide an algebraic characterization of the theory of conjunction and negation common to such views.

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1 Boolean Algebras

We begin with a non-standard axiomatization of Boolean algebras, which will facilitate comparison with the more general class of algebras that are the topic of this paper.

Definition 1 A *Boolean algebra* is a tuple $\mathbf{B} = \langle B, \wedge, \neg, \top \rangle$ where B is a non-empty set, \wedge is a binary operation on B , \neg is a unary operation on B , $\top \in B$, and the following five implicitly universally quantified axioms hold:

COMMUTATIVITY

$$p \wedge q = q \wedge p$$

DISTRIBUTION

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

IDENTITY

$$p \wedge \top = p$$

CONTRADICTION

$$p \wedge \neg \top = p \wedge \neg p$$

EQUIVALENCE

$$\text{If } p \equiv_L q, \text{ then } p = q.$$

where \vee and \equiv_L are defined as follows:

$$p \vee q := \neg(\neg p \wedge \neg q)$$

$$p \equiv_L q := p \wedge q = p \vee q$$

Proposition 2 The above five axioms imply the familiar Boolean identities:

ASSOCIATIVITY

$$p \wedge (q \wedge r) = (p \wedge q) \wedge r$$

IDEMPOTENCE

$$p \wedge p = p$$

INVOLUTION

$$\neg \neg p = p$$

ANNIHILATION

$$p \vee \neg p = \top$$

ABSORPTION

$$p \wedge (p \vee q) = p$$

This and other non-obvious facts asserted without proof have been verified using Prover9 and Mace4 software; see [7].

Conjecture 3 The above five axioms are mutually independent.

Evidence: I have verified the independence of DISTRIBUTION, CONTRADICTION, and EQUIVALENCE using Mace4; the open question concerns COMMUTATIVITY and IDENTITY. My inability to find a proof of either axiom from the other four axioms

using Prover9 suggests their independence; my inability to establish their independence using Mace4 suggests that both are implied by the other four axioms over the class of finite models; Branden Fitelson (p.c.) has established this implication over the class of models of cardinality ≤ 8 .¹

2 Agglomerative Algebras

We now turn to the class of algebras that are the topic of this paper.

Definition 4 An *agglomerative algebra* is a tuple $\mathbf{A} = \langle A, \wedge, \neg, \top \rangle$ where A is a non-empty set, \wedge is a binary operation on A , \neg is a unary operation on A , $\top \in A$, and the following five implicitly universally quantified axioms hold:

COMMUTATIVITY

$$p \wedge q = q \wedge p$$

DISTRIBUTION

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

IDENTITY

$$p \wedge \top = p$$

CONTRADICTION

$$p \wedge \neg \top = p \wedge \neg p$$

TRANSITIVITY

$$\text{If } p \equiv_L q \text{ and } q \equiv_L r, \text{ then } p \equiv_L r.$$

¹Here is a strategy for establishing the independence of IDENTITY. Let S be the set of sentences of the propositional language with signature $\{\wedge, \neg, \top\}$. For $X, Y \in S$, say that X and Y are *primitively convertible* just in case, for some $A, B, C \in S$, either

$$\begin{aligned} \{X, Y\} &= \{\ulcorner A \wedge B \urcorner, \ulcorner B \wedge A \urcorner\}, \text{ or} \\ \{X, Y\} &= \{\ulcorner A \wedge \neg(\neg B \wedge \neg C) \urcorner, \ulcorner \neg(\neg(A \wedge B) \wedge \neg(A \wedge C)) \urcorner\}, \text{ or} \\ \{X, Y\} &= \{\ulcorner A \wedge \neg \top \urcorner, \ulcorner A \wedge \neg A \urcorner\}. \end{aligned}$$

Say that X and Y are *immediately convertible* just in case Y is the result of substituting an occurrence of A for an occurrence of B in X , for some primitively convertible A and B . Let \approx be the transitive reflexive closure of being immediately convertible. \approx is a congruence with respect to negation and conjunction, and when these operations are lifted to S/\approx the result is a model of COMMUTATIVITY, DISTRIBUTION, and CONTRADICTION, by construction, but not a model of IDENTITY, since $\top \not\approx \ulcorner \top \wedge \top \urcorner$. I don't know whether it is a model of EQUIVALENCE, but I conjecture that it is – and, more strongly, that $\ulcorner A \wedge B \urcorner \not\approx \ulcorner \neg(\neg(A \wedge \neg B)) \urcorner$ for all A, B .

A similar strategy could be tried to establish the independence of COMMUTATIVITY, by replacing the first clause in the definition of primitive convertibility with

$$\{X, Y\} = \{A, \ulcorner A \wedge \top \urcorner\}.$$

COMMUTATIVITY is invalid when the definition of \approx is modified accordingly, since $\ulcorner \top \wedge \neg \top \urcorner \not\approx \ulcorner \neg \top \wedge \top \urcorner$. The open question is again whether EQUIVALENCE comes out valid, and I again conjecture that it does (and vacuously so).

This characterization of agglomerative algebras differs from the above characterization of Boolean algebras in that EQUIVALENCE has been weakened to TRANSITIVITY. We begin by establishing some parallels with Boolean algebras.

Proposition 5 The above five axioms are mutually independent.

Proposition 6 The axioms of agglomerative algebras imply ASSOCIATIVITY, IDEMPOTENCE, and INVOLUTION.

However, not every agglomerative algebra is a Boolean algebra. Consider $\mathbf{A}_{2 \times 2} = \langle A, \wedge, \neg, \top \rangle$ where $A = \{0, 1\} \times \{0, 1\}$, $\wedge : \langle \langle x, y \rangle, \langle x', y' \rangle \rangle \mapsto \langle x \cdot x', y \cdot y' \rangle$, $\neg : \langle x, y \rangle \mapsto \langle 1 - x, y \rangle$, and $\top = \langle 1, 1 \rangle$.

Proposition 7 $\mathbf{A}_{2 \times 2}$ is an agglomerative algebra but is not a Boolean algebra.

Proof: It is straightforward to verify that $\mathbf{A}_{2 \times 2}$ is an agglomerative algebra. But $\langle 1, 0 \rangle \vee \neg \langle 1, 0 \rangle = \langle 1, 0 \rangle \neq \top$, violating ANNIHILATION, and $\top \wedge (\top \vee \langle 1, 0 \rangle) = \langle 1, 0 \rangle \neq \top$, violating ABSORPTION.

The guiding idea behind agglomerative algebras is that a proposition's identity is jointly determined by its logical content and its non-logical contents. We will now make this idea precise. (In what follows quantification over agglomerative algebras $\mathbf{A} = \langle A, \wedge, \neg, \top \rangle$ will be left implicit, with p, q , and r understood as ranging over A .)

Proposition 8 \equiv_L is an equivalence relation on A .

The relation \equiv_L should be thought of as expressing sameness of logical content. We now define the relation of sameness of non-logical content.

Definition 9 $p \equiv_N q := p \vee \neg p = q \vee \neg q$

Proposition 10 \equiv_N is an equivalence relation on A .

Distinct members of A must either have different logical contents or different non-logical contents.

Proposition 11 If $p \equiv_L q$ and $p \equiv_N q$, then $p = q$.

Note that neither \equiv_L nor \equiv_N need suffice for identity.

Remark 12 In $\mathbf{A}_{2 \times 2}$, $\langle 1, 1 \rangle \equiv_L \langle 1, 0 \rangle$ and $\langle 1, 1 \rangle \equiv_N \langle 0, 1 \rangle$.

We can now make precise the sense in which the elements of an agglomerative algebra have logical contents that form a Boolean algebra with respect to conjunction and negation.

- Definition 13** (i) $[p]_L := \{q : q \equiv_L p\}$
 (ii) $A/\equiv_L := \{[p]_L : p \in A\}$
 (iii) $\neg_L [p]_L := [\neg p]_L$
 (iv) $[p]_L \wedge_L [q]_L := [p \wedge q]_L$
 (v) $\mathbf{A}_L := \langle A/\equiv_L, \wedge_L, \neg_L, [\top]_L \rangle$

\neg_L and \wedge_L are well defined because \equiv_L is a congruence with respect to \neg and \wedge in the following sense:

Proposition 14 $\neg p \equiv_L \neg p'$ whenever $p \equiv_L p'$, and $p \wedge q \equiv_L p' \wedge q'$ whenever $p \equiv_L p'$ and $q \equiv_L q'$.

We now establish two lemmas (where $x \vee_L y := \neg_L(\neg_L x \wedge_L \neg_L y)$).

Lemma 15 $[p]_L \vee_L \neg_L [p]_L = [\top]_L$.

Lemma 16 Any agglomerative algebra that satisfies ANNIHILATION is Boolean.

Proposition 17 \mathbf{A}_L is a Boolean algebra.

Proof: \equiv_L a congruence with respect to \neg and \wedge , so the lifted operations \neg_L and \wedge_L satisfy the axioms of agglomerative algebras. And they also satisfy ANNIHILATION. So they must satisfy the axioms of Boolean algebras too.

Corollary 18 An agglomerative algebra is Boolean just in case \equiv_N is a universal relation on A .

Proposition 19 Any agglomerative algebra that satisfies ABSORPTION is Boolean.

Corollary 20 An agglomerative algebra is Boolean just in case it is a lattice with respect to \wedge and \vee .

We now turn to non-logical content.

- Definition 21** (i) $[p]_N := \{q : q \equiv_N p\}$
 (ii) $A/\equiv_N := \{[p]_N : p \in A\}$
 (iii) $\neg_N [p]_N := [\neg p]_N$
 (iv) $[p]_N \wedge_N [q]_N := [p \wedge q]_N$
 (v) $\mathbf{A}_N := \langle A/\equiv_N, \wedge_N, \neg_N, [\top]_N \rangle$

\neg_N and \wedge_N are well-defined because \equiv_N is a congruence with respect to \neg and \wedge , in the sense that:

Proposition 22 $\neg p \equiv_N \neg p'$ whenever $p \equiv_N p'$, and $p \wedge q \equiv_N p' \wedge q'$ whenever $p \equiv_N p'$ and $q \equiv_N q'$.

We can now make precise the claims that non-logical content is undisturbed by negation and that it agglomerates under conjunction, with \top having minimal non-logical content.

Proposition 23 $[p]_N = \neg_N[p]_N$

Proposition 24 \mathbf{A}_N is a bounded meet-semilattice (i.e., a set together with a commutative, associative, idempotent operation and a distinguished identity element).

3 Representability

In this section we prove that the class of agglomerative algebras is isomorphic to the class of subalgebras of direct products of Boolean algebras and bounded meet-semilattices.

Definition 25 A *concrete agglomerative algebra* is a tuple $\mathbf{C} = \langle C, \wedge, \neg, \top \rangle$ where C is a non-empty set, \wedge is a binary operation on C , \neg is a unary operation on C , $\top \in C$, and for some \mathbf{L} and \mathbf{N} (on which we say \mathbf{C} is *based*)

- (i) $\mathbf{L} = \langle L, \wedge_L, \neg_L, \top_L \rangle$ is a Boolean algebra;
- (ii) $\mathbf{N} = \langle N, \wedge_N, \top_N \rangle$ is a bounded meet-semilattice;
- (iii) $C \subseteq L \times N$;
- (iv) $\langle l, n \rangle \wedge \langle l', n' \rangle = \langle l \wedge_L l', n \wedge_N n' \rangle$;
- (v) $\neg \langle l, n \rangle = \langle \neg_L l, n \rangle$;
- (vi) $\top = \langle \top_L, \top_N \rangle$.

Proposition 26 Concrete agglomerative algebras are agglomerative algebras.

Definition 27 Where $\mathbf{A} = \langle A, \wedge, \neg, \top \rangle$ is an agglomerative algebra, let $\mathbf{A}^* := \langle A^*, \wedge^*, \neg^*, \top^* \rangle$ where

- (i) $A^* = \{ \langle [p]_L, [p]_N \rangle : p \in A \}$;
- (ii) $\langle [p]_L, [p]_N \rangle \wedge^* \langle [q]_L, [q]_N \rangle = \langle [p]_L \wedge_L [q]_L, [p]_N \wedge_N [q]_N \rangle$;
- (iii) $\neg^* \langle [p]_L, [p]_N \rangle = \langle \neg_L [p]_L, \neg_N [p]_N \rangle$;
- (iv) $\top^* = \langle [\top]_L, [\top]_N \rangle$.

Lemma 28 \mathbf{A} is isomorphic to \mathbf{A}^* .

Proof sketch: $f : p \mapsto \langle [p]_L, [p]_N \rangle$ is an isomorphism.

Lemma 29 \mathbf{A}^* is a concrete agglomerative algebra.

Proof sketch: Let $\mathbf{L} = \mathbf{A}_L$ and $\mathbf{N} = \mathbf{A}_N$.

Proposition 30 Every agglomerative algebra is isomorphic to a concrete agglomerative algebra.

It is often helpful to have a representation of algebraic structures in terms of fields of sets. To this end, we define the following class of structures.

Definition 31 A *powerset agglomerative algebra* is a tuple $\mathbf{P} = \langle P, \wedge, \neg, \top \rangle$ where P is a non-empty set, \wedge is a binary operation on P , \neg is a unary operation on P , $\top \in P$, and for some sets W and V (on which we say \mathbf{P} is *based*)

- (i) $P \subseteq \mathcal{P}(W) \times \mathcal{P}(V)$;
- (ii) $\langle X, Y \rangle \wedge \langle X', Y' \rangle = \langle X \cap X', Y \cup Y' \rangle$;
- (iii) $\neg \langle X, Y \rangle = \langle W \setminus X, Y \rangle$;
- (iv) $\top = \langle W, \emptyset \rangle$.

Proposition 32 Powerset agglomerative algebras are agglomerative algebras.

Proposition 33 Every agglomerative algebra is isomorphic to a powerset agglomerative algebra.

Proof sketch: Stone's representation theorem for Boolean algebras and the parallel representation theorem for semilattices immediately yield a powerset agglomerative algebra isomorphic to any given concrete agglomerative algebra.

4 Orders on Agglomerative Algebras

Definition 34 $p \leq q := p = p \wedge q$; $p \leq^* q := q = p \vee q$

Proposition 35 Both \leq and \leq^* are partial orders.

Proposition 36 \leq and \leq^* are coextensive if and only if \mathbf{A} is a Boolean algebra.

Proposition 37 Agglomerative algebras can be isomorphic with respect to \leq and isomorphic with respect to \leq^* without being isomorphic.

Proof: For Boolean algebras $\mathbf{B}_1 = \langle B_1, \wedge_1, \neg_1, \top_1 \rangle$ and $\mathbf{B}_2 = \langle B_2, \wedge_2, \neg_2, \top_2 \rangle$, let $\mathbf{A}_{\mathbf{B}_1\mathbf{B}_2} = \langle A, \wedge, \neg, \top \rangle$ be the concrete agglomerative algebra such that $A = B_1 \times B_2$, $\langle x_1, x_2 \rangle \wedge \langle y_1, y_2 \rangle = \langle x_1 \wedge_1 y_1, x_2 \wedge_2 y_2 \rangle$, $\neg \langle x_1, x_2 \rangle = \langle \neg_1 x_1, x_2 \rangle$, and $\top = \langle \top_1, \top_2 \rangle$. Suppose $\mathbf{B}_1 \not\cong \mathbf{B}_2$. Then $\mathbf{A}_{\mathbf{B}_1\mathbf{B}_2} \not\cong \mathbf{A}_{\mathbf{B}_2\mathbf{B}_1}$. But they are isomorphic with respect to \leq , as witnessed by $f : \langle x, y \rangle \mapsto \langle y, x \rangle$, and isomorphic with respect to \leq^* , as witnessed by $g : \langle x, y \rangle \mapsto \langle \neg_2 y, \neg_1 x \rangle$.

Proposition 38 Agglomerative algebras are isomorphic whenever there is a bijection between them that preserves both \leq and \leq^* .

Proof sketch: We define \wedge as the greatest lower bound under \leq and \vee as the least upper bound under \leq^* . We identify \top as the element satisfying IDENTITY. Say that p is a *contradiction* just in case, for all q , $p \wedge q \equiv_L p$. We identify \neg with the operation mapping every p to the q such that $p \wedge q$ is a contradiction, $p \vee q \equiv_L \top$, and $p \vee \top = q \vee \top$.

Definition 39 \mathbf{A} is *complete* $:=$ for all $X \subseteq A$, there is some $x \in A$ – which we denote $\bigwedge X$ – such that x is the greatest lower bound of X under \leq .

Definition 40

$$x \leq_L y := x = x \wedge_L y$$

$$x \leq_N y := x = x \wedge_N y$$

Proposition 41 If \mathbf{A} is complete, then \mathbf{A}_L is a complete Boolean algebra and \mathbf{A}_N is a complete bounded meet-semilattice.

Proof sketch: For $X \subseteq A/\equiv_L$, $[\bigwedge([\bigwedge A]_N \cap \bigcup X)]_L$ is the greatest lower bound of X under \leq_L ; for $X \subseteq A/\equiv_N$, $[\bigwedge([\top]_L \cap \bigcup X)]_N$ is the greatest lower bound of X under \leq_N .

Remark 42 That \mathbf{A}_L is a complete Boolean algebra and \mathbf{A}_N a complete bounded meet-semilattice does not imply that \mathbf{A} is complete.

Proof: Let $\mathbf{L} = \langle \mathcal{P}(\mathbb{R}), \cup, \mathbb{R} \setminus \cdot, \mathbb{R} \rangle$ and $\mathbf{N} = \langle \{0, 1, \dots, \omega, \omega + 1\}, \max(\cdot, \cdot), 0 \rangle$. Now consider the concrete agglomerative algebra \mathbf{A} based on \mathbf{L} and \mathbf{N} such that $A = (I \times \{0, 1, \dots, \omega\}) \cup (\mathcal{P}(\mathbb{R}) \times \{\omega + 1\})$, where I is the closure of $\{[x, y] \subseteq \mathbb{R} : x < y\}$ under finite intersections and complementation with \mathbb{R} . It is easy to verify that \mathbf{A}_L is isomorphic to \mathbf{L} , and hence is a complete Boolean algebra, and that \mathbf{A}_N is isomorphic to \mathbf{N} , and hence is a complete bounded meet-semilattice. Now consider $X = \{[0, 1/n], n\} : 0 < n < \omega\}$. Both $\langle \emptyset, \omega \rangle$ and $\langle \{0\}, \omega + 1 \rangle$ are lower bounds of X under \leq . But there is no lower bound of X that they are both \leq . So there is no greatest lower bound of X under \leq , and hence \mathbf{A} is not complete.

Proposition 43 If \mathbf{A} is complete and $X \subseteq A$, then $[\bigwedge X]_L$ is the greatest lower bound of $\{[x]_L : x \in X\}$ under \leq_L and $[\bigwedge X]_N$ is the greatest lower bound of $\{[x]_N : x \in X\}$ under \leq_N .

Proof sketch: Let l be the greatest lower bound of $\{[x]_L : x \in X\}$ under \leq_L and n be the greatest lower bound of $\{[x]_N : x \in X\}$ under \leq_N . Clearly $[\bigwedge X]_L \leq_L l$ and $[\bigwedge X]_N \leq_N n$, since \equiv_L and \equiv_N are both congruences with respect to \wedge . And $l \leq_L [\bigwedge X]_L$, since the element in $l \cap [\bigwedge A]_N$ is a lower bound of X under \leq , and so must be $\leq \bigwedge X$; so $[\bigwedge X]_L = l$. Similarly, $n \leq_N [\bigwedge X]_N$, since the element in $[\neg \top]_L \cap n$ is a lower bound of X under \leq , and so must be $\leq \bigwedge X$; so $[\bigwedge X]_N = n$.

Proposition 44 If \mathbf{A} is complete, then for all $X \subseteq A$, there is some $x \in A$ – which we denote $\bigvee X$ – such that x is the least upper bound of X under \leq^* .

Proof sketch: $\bigvee \emptyset = \bigwedge A$. For $X \neq \emptyset$, it suffices show that, for some x , $[x]_L$ is the least upper bound of $\{[y]_L : y \in X\}$ under \leq_L^* and of $\{[y]_N : y \in X\}$ under \leq_N^* . The latter condition is equivalent to $[x]_N$ being the greatest upper bound of $\{[y]_N : y \in X\}$ under \leq_N (since \leq_N^* is the converse of \leq_N), which it is just in case $y \equiv_N \bigwedge X$. Our earlier results imply that $\langle \{[y]_L : y \equiv_N \bigwedge X\}, \wedge_L, \neg_L, [\top]_L \rangle$ is a complete Boolean subalgebra of \mathbf{A}_L . So there is some $x \equiv_N \bigwedge X$ such that $[x]_L$ is the least upper bound of $\{[y]_L : y \in X\}$ under \leq_L , and hence also under \leq_L^* , since these relations coincide in Boolean algebras.

Remark 45 $p \wedge q$ is the greatest lower bound of p and q under \leq , and $p \vee q$ is the least upper bound of p and q under \leq^* . But $p \vee q$ need not be the least upper bound of p and q under \leq ; indeed, it will be only if $p \equiv_N q$. In fact, p and q need not have any least upper bound under \leq , and even if they do, it may be neither $\equiv_L p \vee q$ nor $\equiv_N p \vee q$.

Proof sketch: Let $\mathbf{L} = \langle \mathcal{P}(W), \cap, W \setminus \cdot, W \rangle$ where $W = \{0, 1, 2\}$. Let $\mathbf{N} = \langle N, \cup, 0 \rangle$ where $N = \omega \cup \{a, a', \omega + 2\}$, $a = (\omega + 2) \setminus \{\omega\}$, $a' = (\omega + 2) \setminus \{\omega + 1\}$. Let \mathbf{A} be the concrete agglomerative algebra based on \mathbf{L} and \mathbf{N} such that $A = \bigcup \{ \{W, \emptyset\} \times \omega, \{W, \{0\}, \{1, 2\}, \emptyset\} \times \{a\}, \{W, \{0, 1\}, \{2\}, \emptyset\} \times \{a'\}, \mathcal{P}(W) \times \{\omega + 2\} \}$. Notice that $\langle \{0\}, a \rangle$ and $\langle \{2\}, a' \rangle$ have no least upper bound under \leq . Now replace A with $A \setminus \{ \langle l, n \rangle \in A : 0 < n < \omega \}$. $\langle \{0\}, a \rangle$ and $\langle \{2\}, a' \rangle$ now do have a least upper bound under \leq , namely $\langle W, 0 \rangle$, but it is neither \equiv_L nor \equiv_N their disjunction $\langle \{0, 2\}, \omega + 2 \rangle$.

We may summarize the situation regarding completeness as follows. An agglomerative algebra is complete insofar as we can think of it as being closed under the formation of infinite conjunctions and disjunctions, understood in terms of the existence of greatest lower bounds under the conjunctively defined order \leq and the existence of least upper bounds under the disjunctively defined order \leq^* . These conjunctive and disjunctive conditions are equivalent. Provided they hold, we can think of infinite conjunction as operating component-wise on logical contents and on non-logical contents, just as we can in the case of binary conjunction. However, we must be careful to distinguish \leq from \leq^* , since \leq does not reflect disjunctive structure and \leq^* does not reflect conjunctive structure.

Definition 46 p is an *atom* := there is a unique $q \neq p$ such that $q \leq p$.

Definition 47 p is *maximally consistent* := for all q , $p \leq q$ iff $p \not\leq \neg q$.

These two notions coincide in Boolean algebras, but not in agglomerative algebras.

Proposition 48 Being an atom does not imply being maximally consistent.

Proof: In $\mathbf{A}_{2 \times 2}$, $\langle 0, 1 \rangle$ is an atom. But it is not maximally consistent, since $\langle 0, 1 \rangle \leq \langle 0, 1 \rangle$ and $\langle 0, 1 \rangle \leq \neg \langle 0, 1 \rangle = \langle 1, 1 \rangle$.

Proposition 49 Being maximally consistent does imply being an atom.

Proof sketch: If p is maximally consistent, then p has maximally consistent logical content and maximal non-logical content, in which case there is exactly one $q \neq p$ such that $q \leq p$: the contraction with maximal non-logical content.

Despite the duality of \wedge and \vee in agglomerative algebras, it is usually more convenient to theorize in terms of \leq than in terms of \leq^* , since maximally consistent propositions (such as the conjunction of all truths) tend to be easier to think about than the dual notion of minimally non-trivial propositions (such as the disjunction of all falsehoods). Indeed, for many applications we can think of maximally consistent elements of an agglomerative algebra as corresponding to possible worlds in the familiar way.

It is well known that complete atomic Boolean algebras with the same number of atoms are isomorphic. This is not the case for agglomerative algebras, even provided that they generate isomorphic algebras of logical contents, isomorphic algebras of non-logical contents, and are of the same cardinality.

Example 50 Let \mathbf{A} and \mathbf{A}' be powerset agglomerative algebras, both based on $W = V = \{0, 1, 2\}$, where $A =$

$$\begin{aligned} &(\{\emptyset, W\} \times \{\emptyset, \{0\}, \{1\}, \{2\}\}) \cup \\ &(\{\emptyset, \{0\}, \{1, 2\}, W\} \times \{\{0, 1\}, \{0, 2\}, \{1, 2\}\}) \cup \\ &(\mathcal{P}(W) \times \{V\}) \end{aligned}$$

and $A' =$

$$\begin{aligned} &(\{\emptyset, W\} \times \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}) \cup \\ &(\{\emptyset, \{0\}, \{1, 2\}, W\} \times \{\{2\}, \{0, 2\}, \{1, 2\}\}) \cup \\ &(\mathcal{P}(W) \times \{V\}) \end{aligned}$$

\mathbf{A} and \mathbf{A}' are complete and atomic, $\mathbf{A}_L \cong \mathbf{A}'_L$, $\mathbf{A}_N \cong \mathbf{A}'_N$, $|A| = |A'| = 28$ and they have the same three atoms $\langle\{0\}, W\rangle$, $\langle\{1\}, W\rangle$, and $\langle\{2\}, W\rangle$. But $\mathbf{A} \not\cong \mathbf{A}'$.

5 Parry Algebras

We will here consider the generalization of agglomerative algebras that results from relaxing the requirement that the semilattice of non-logical contents be bounded. Let a *concrete Parry algebra* be a triple $\mathbf{X} = \langle X, \wedge, \neg \rangle$ where X is a non-empty set, \wedge is a binary operation on X , \neg is a unary operation on X , and for some \mathbf{L} and \mathbf{N} (on which we say \mathbf{X} is *based*)

- (i) $\mathbf{L} = \langle L, \wedge_{\mathbf{L}}, \neg_{\mathbf{L}}, \top_{\mathbf{L}} \rangle$ is a Boolean algebra;
- (ii) $\mathbf{N} = \langle N, \wedge_{\mathbf{N}} \rangle$ is a semilattice;
- (iii) $X \subseteq L \times N$;
- (iv) $\langle l, n \rangle \wedge \langle l', n' \rangle = \langle l \wedge_{\mathbf{L}} l', n \wedge_{\mathbf{N}} n' \rangle$;
- (v) $\neg \langle l, n \rangle = \langle \neg_{\mathbf{L}} l, n \rangle$.

Let a *Parry algebra* be any algebraic structure that is isomorphic to a concrete Parry algebra. By our earlier representation theorems, if $\langle A, \wedge, \neg, \top \rangle$ is an agglomerative algebra, then $\langle A, \wedge, \neg \rangle$ is a Parry algebra. But not every Parry algebra $\langle A, \wedge, \neg \rangle$ can be extended to an agglomerative algebra $\langle A, \wedge, \neg, \top \rangle$: consider the direct product of any Boolean algebra with any non-bounded meet-semilattice. This class of structures is named after Parry's [8, 9] logic of analytic implication, which can be reinterpreted as the view that propositions form a Parry algebra by interpreting his conditional as expressing \leq . (The connection to Boolean algebras and semilattices is exhibited by the models given by Fine in [5], and both Parry and Fine give a gloss on the system not unlike my gloss in terms of the accumulation of non-logical content; see also [3, chapter 1].)

Proposition 51 The class Parry algebras is axiomatized by the universal closures of ASSOCIATIVITY, IDEMPOTENCE, DISTRIBUTION, TRANSITIVITY,

PARRITY

$$p \wedge (q \wedge \neg q) = q \wedge (p \wedge \neg p)$$

and

INDIVIDUATION

$$\text{If } p \equiv_L q \text{ and } p \equiv_N q, \text{ then } p = q.$$

Proposition 52 These axioms are mutually independent.

6 Connections to Universal Algebra

Definition 53 A *variety* is a class of algebras of a given type definable by a set of universally quantified equations.

Proposition 54 The class of agglomerative algebras is not a variety.

Proof: Consider $\mathbf{WK} = \langle \{0, \frac{1}{2}, 1\}, \wedge, \neg, 1 \rangle$ where $\neg x = -x$ and $x \wedge y = \frac{1}{2}$ if either $x = \frac{1}{2}$ or $y = \frac{1}{2}$ and $x \cdot y$ otherwise; it is isomorphic to the weak Kleene truth table. \mathbf{WK} is not an agglomerative algebra, since $0 \equiv_L \frac{1}{2}$ and $\frac{1}{2} \equiv_L 1$, but $0 \not\equiv_L 1$, violating TRANSITIVITY. Now observe that the map $f : \mathbf{A}_{2 \times 2} \rightarrow \mathbf{WK} = \{ \langle \langle 0, 1 \rangle, 0 \rangle, \langle \langle 0, 0 \rangle, \frac{1}{2} \rangle, \langle \langle 1, 0 \rangle, \frac{1}{2} \rangle, \langle \langle 1, 1 \rangle, 1 \rangle \}$ is a surjective homomorphism. (I owe this observation to Harvey Lederman.) So the class of agglomerative algebras is not closed under homomorphic images. So it is not a variety, by Birkhoff's HSP theorem.

Definition 55 A *quasiidentity* is an implication whose antecedent is a conjunction of equations and whose consequent is an equation; a *quasivariety* is a class of algebras definable by a set of universally quantified quasiidentities.

Proposition 56 The class of agglomerative algebras is a quasivariety, and so includes a trivial algebra and is closed under isomorphisms, subalgebras, direct products, and ultraproducts.

Proof: The latter condition is equivalent to being a quasivariety by Mal'cev's analogue for quasivarieties of Birkhoff's HSP theorem for varieties.

Definition 57 A *regular equation* is one in which the same variables occur on both sides; a *regular quasiidentity* is a quasiidentity containing only regular equations.

Proposition 58 The class of agglomerative algebras is defined by the set of regular quasiidentities that are valid on the class of Boolean algebras.

Proof sketch: [1, Theorem 3.5 and Example 7.11] in effect prove this result for Parry algebras in relation to Boolean algebras without the nullary polynomial \top ; the generalization to agglomerative algebras is a straightforward application of their proof once allowance is made for nullary polynomials as in [10].

[1] contains a number of further results that illuminate the connection between agglomerative and Boolean algebras, by investigating more generally the properties of regular quasivarieties that are intermediate between a given variety and its regularization (the variety defined by the regular identities satisfied in it). For example, their Proposition 3.3 together with the above result implies that the class of agglomerative algebras is identical to the class of Płonka sums of families of Boolean algebras indexed by bounded semilattices where all Płonka-homomorphisms are injective. (I only became aware of these results in the final stage of preparing this manuscript for production, or else I would have discussed them at greater length.)

7 Two Interpretations

We will now consider two concrete proposals about the metaphysics of propositions according to which they form a non-Boolean agglomerative algebra.

7.1 Partition Agglomerative Algebras

A natural idea suggested by (although not made in) recent work by Yablo [12] and and Yalcin [13, 14] is that propositions true in the same possible worlds can be distinct by differing in *subject matter* – where, following [6], we identify subject matters with partitions of the space of possible worlds, with two worlds in the same cell of the partition just in case they agree on the subject matter in question. On this view, propositions do not form a Boolean algebra, but they do form an agglomerative algebra.

Definition 59 A *partition agglomerative algebra* is a concrete agglomerative algebra based on a Boolean algebra $\mathbf{L} = \langle \mathcal{P}(W), \cap, W \setminus \cdot, W \rangle$ and bounded meet-semilattice $\mathbf{N} = \langle \Pi(W), \wedge_\Pi, \{W\} \rangle$, where W is a non-empty set, $\Pi(W)$ is the set of partitions

of W , $x \wedge_{\Pi} y = \{p \cap q : p \in x, q \in y\} \setminus \{\emptyset\}$, and for all $\langle X, \pi \rangle \in A$, π admits X (in the sense that $X = \bigcup S$ for some $S \subseteq \pi$).

Proposition 60 An agglomerative algebra is isomorphic to a partition agglomerative algebra just in case, for all p , if $p \equiv_N \top$, then $p = \top$ iff $p \neq \neg \top$.

Proof: Left to right: immediate from the fact that $\{W\}$ admits only W and \emptyset , which are distinct since $\{W\}$ is a partition. Right to left: By our earlier representation theorems, we can restrict our attention to powerset agglomerative algebras. Let $\mathbf{P} = \langle P, \wedge, \neg, \top \rangle$ be a powerset algebra based on sets W and V ; we may stipulate that $W \cap V = \emptyset$. Given an arbitrarily chosen $w \in W$ (the existence of which is guaranteed by the distinctness of \top and $\neg \top$) and $a \notin (W \cup V)$, let $W' = W \cup V \cup \{a\}$ and define injections $f : \mathcal{P}(W) \rightarrow \mathcal{P}(W')$ and $g : \Pi(V) \rightarrow \Pi(W')$ as follows:

$$\begin{aligned} f(X) &= X \cup V \cup \{a\} \text{ if } w \in X \text{ and } = X \text{ otherwise} \\ g(Y) &= \{W'\} \text{ if } Y = V \text{ and } = \{W' \setminus (W \cup Y)\} \cup \{x\} : x \in (W \cup Y) \text{ otherwise} \end{aligned}$$

It is straightforward to verify that \mathbf{P} is isomorphic to the partition agglomerative algebra with carrier set $\{\langle f(X), g(Y) \rangle : \langle X, Y \rangle \in P\}$ based on $\langle \mathcal{P}(W'), \cap, W' \setminus \cdot, W' \rangle$ and $\langle \Pi(W'), \wedge_{\Pi}, \{W'\} \rangle$.

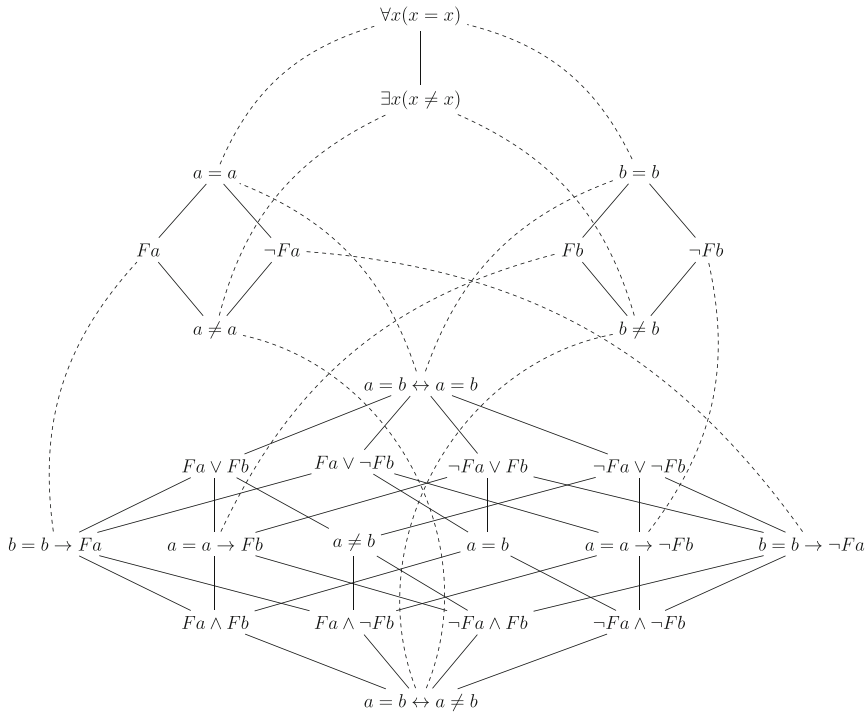
7.2 Aboutness

A different reason one might think that propositions form a non-Boolean agglomerative algebra is the idea that logically equivalent propositions can be distinct by differing in which entities they are *about*. Agglomerative algebras offer a tool for endorsing this sort of hyperintensionality without resorting to the idea of propositions as structured entities, and thereby offer a way to avoid the pitfalls and paradoxes associated with structured propositions. Here are two ways in which we might model this idea of unstructured entity-involving propositions.

Example 61 Think of propositions as equivalence classes of sentences under the relation of being both logically equivalent and containing the same relevant constants. Formally, let \mathcal{L} be a finitary first-order language with identity, let C be a set of non-logical constants of \mathcal{L} , and let S be the set of sentences of \mathcal{L} . Let $\varphi \sim \psi := \models \varphi \leftrightarrow \psi$ and, for all $c \in C$, c occurs in φ if and only if c occurs in ψ . Now define $[\varphi] := \{\psi : \psi \sim \varphi\}$; $A/\sim := \{[\varphi] : \varphi \in S\}$; $\neg \sim : [\varphi] \mapsto [\neg \varphi]$; $\wedge \sim : [\varphi], [\psi] \mapsto [\varphi \wedge \psi]$; and $\top \sim := [\neg \forall x(x = x)]$. ($\neg \sim$ and $\wedge \sim$ are well defined because $\neg \varphi \sim \neg \varphi'$ whenever $\varphi \sim \varphi'$, and $\varphi \wedge \psi \sim \varphi' \wedge \psi'$ whenever $\varphi \sim \varphi'$ and $\psi \sim \psi'$.) $\mathbf{A} = \langle A/\sim, \wedge \sim, \neg \sim, \top \sim \rangle$ is then an agglomerative algebra, with \mathbf{A}_L isomorphic to the Lindenbaum-Tarski algebra of \mathcal{L} (i.e., the structure defined like \mathbf{A} except without the requirement that φ and ψ contain the same constants in C) and $\mathbf{A}_N \cong \langle \{X \subseteq C : |X| < \omega\}, \cup, \emptyset \rangle$.

We can use this technique to generate finite algebras by defining \models as truth in every model of \mathcal{L} in a given class that contains only finitely many models (up to isomorphism). For example, if $C = \{a, b\}$ and \mathcal{L} contains one monadic predicate constant F , then by defining \models as truth in all models of cardinality 2 in which one

thing is in the extension of F we generate the 26-element agglomerative algebra depicted in the Hasse diagram below, in which lines denote \leq , straight solid lines denote \equiv_N , dashed curved lines denote \equiv_L , and brackets and corner quotes for are omitted for readability.



Remark 62 In a partition agglomerative algebra, if $[p]_N = n_1 \wedge_N n_2$, then $p = \bigvee \{q \wedge r : \langle q, r \rangle \in X\}$ for some $X \subseteq n_1 \times n_2$. By contrast, this is not true in the above example, since $\lceil a = b \rceil$ is a counterexample. This fact shows that if the subject matter of a proposition is thought of not as a partition of logical space, but instead as the set of entities that the proposition is about, then we cannot assume that a proposition whose subject matter is the combination of two other subject matters will be a Boolean combination of propositions with one of the two other subject matters.

Example 63 Think of propositions as forming a powerset agglomerative algebra based on the set of possible worlds W and of individuals V , where $\langle X, Y \rangle$ is true at all and only the worlds in X and is about all and only the individuals in Y . Formally, let S_W be the group of all permutations of W , let G be a group of permutations of V , let $\text{fix}(Y) := \{g \in G : g(y) = y \text{ for all } y \in Y\}$, let $f : G \rightarrow S_W$ be a group homomorphism, and let $A = \{\langle X, Y \rangle : w \in X \text{ if and only if } f(g)(w) \in X, \text{ for all } g \in \text{fix}(Y) \text{ and } w \in W\}$. It is straightforward to verify that A is the carrier set of a powerset agglomerative algebra \mathbf{A} based on W and V , with $\mathbf{A}_L \cong$

$\langle \mathcal{P}(W), \cap, W \setminus \cdot, W \rangle$ and $\mathbf{A}_N \cong \langle \mathcal{P}(V), \cup, \emptyset \rangle$. Intuitively, think of G as the group of all permutations of individuals g that determine permutations of possible worlds $f(g)$, in the sense that y_1, y_2, \dots stand in the same qualitative relations at w as $g(y_1), g(y_2), \dots$ do at $f(g)(w)$, for any world w and individuals y_1, y_2, \dots . There is such a proposition as $\langle X, Y \rangle$ just in case the logical content X corresponds to the holding of some qualitative relations among members of Y , in the sense that X is mapped to itself by $f(g)$ for all g that map each member of Y to itself. I explore these ideas further in unpublished work; for ideas in a similar spirit, see [4, §VII.B].

7.3 Another Application

I would like to mention one other application of agglomerative algebras to theories of propositional granularity. Caie, Goodman, and Lederman [2] explore views according to which the indiscernibility of identicals can fail for propositions – for example, perhaps the proposition that Superman flies = the proposition that Clark flies, and Lois believes the proposition that Superman flies, but Lois does not believe the proposition that Clark flies. Even if we allow that the property of being believed by Lois can distinguish identical propositions, we might still wish to deny that this can ever happen with purely logical properties: for example, if $p = q$, and p is true, then q must be true too. It turns out that, assuming classical quantification theory and counting higher-order quantifiers among one's logical constants, such theories are trivialized by the assumption that propositions form a Boolean algebra (either with respect to identity or with respect to indiscernibility). By contrast, such theories have a range of non-trivial models in which propositions form an agglomerative algebra.

8 Informal Completeness

8.1 The Boolean precedent

Suppose that propositions form a Boolean algebra with respect to conjunction and negation into which any countable Boolean algebra can be embedded. (This follows, for example, from the view that propositions form a complete atomic Boolean algebra with infinitely many atoms, which in turn follows from the view that propositions true in the same possible worlds are identical and that there are infinitely many possible worlds.) Then there is a precise sense in which the theory of non-trivial Boolean algebras comprises the complete zeroth-order theory of propositional granularity.

Let \mathcal{L} be the quantifier-free language whose logical vocabulary consists of the unary sentential operator \sim (not), the binary sentential operator $\&$ (and), the binary relation symbol $=$ (identity), the unary function symbol \neg (negation), the binary function symbol \wedge (conjunction), and an individual constant \top (the tautology). The non-logical vocabulary of \mathcal{L} consists of a countable stock of constants p, q, r, \dots . Now let \mathcal{L}^\forall be the result of enriching \mathcal{L} with countably many variables x, y, z, \dots .

ranging over propositions, and with quantifiers capable of binding these variables. For any sentence φ of \mathcal{L} , let $\forall\varphi$ be the sentence of \mathcal{L}^\forall that results from φ by uniformly replacing its non-logical constants with free variables and universally closing the resulting formula. Following [11, §3.3], say that φ is *metaphysically universal* just in case $\forall\varphi$ is true on its intended interpretation. Let \mathcal{L}_{MU} be the set of metaphysically universal sentences of \mathcal{L} .

Definition 64 Let \mathcal{B} be the result of adding

$$\begin{array}{l} \text{NON-TRIVIALITY} \\ \top \neq \neg\top \end{array}$$

to the set of all instances of COMMUTATIVITY, DISTRIBUTION, IDENTITY, CONTRADICTION, and EQUIVALENCE, and closing the result under classical logic.

Proposition 65 If propositions form a Boolean algebra (with respect to conjunction and negation) into which all countable Boolean algebras can be embedded, then $\mathcal{L}_{MU} = \mathcal{B}$.

Proof: $\mathcal{L}_{MU} \supseteq \mathcal{B}$ because all instances of the axioms of \mathcal{B} are metaphysically universal and metaphysical universality is closed under classical consequence. To show that $\mathcal{L}_{MU} \subseteq \mathcal{B}$, suppose that $\varphi \notin \mathcal{B}$. By the completeness and Löwenheim-Skolem theorems for zeroth-order languages, there is a countable model \mathcal{M} of \mathcal{B} that falsifies φ . To establish that some *propositions* witness the falsity of $\forall\varphi$, and hence that $\varphi \notin \mathcal{L}_{MU}$, we appeal to the existence of an embedding of \mathcal{M} (thought of as a non-trivial Boolean algebra) into the algebra of propositions.

8.2 Agglomerative Analogues

Definition 66 Let \mathcal{A}^Π be the result of closing the set of all instances of COMMUTATIVITY, DISTRIBUTION, IDENTITY, CONTRADICTION, EQUIVALENCE and

$$\begin{array}{l} \text{TOP TWO} \\ p \equiv_N \top \rightarrow (p = \top \leftrightarrow p \neq \neg\top) \end{array}$$

under classical logic.

Proposition 67 If propositions form an agglomerative algebra in the manner of the theory of subject-matter-enriched sets of possible worlds mentioned in Section 7.1, then $\mathcal{L}_{MU} = \mathcal{A}^\Pi$.

Proof: As in the Boolean case. Propositions 30, 33 and 60 imply that any countable agglomerative algebra satisfying TOP TWO is isomorphic to a partition agglomerative algebra in which W is countable. Since there are infinitely many possible worlds, this means that any countable model of \mathcal{A}^Π (thought of as an agglomerative algebra) is isomorphic to a subalgebra of propositions, assuming propositions are isomorphic to the class of pairs $\langle X, \pi \rangle$ where π is a partition of worlds and X is the union of some subset of π . (If worried that there are too many worlds to form a set, we can get by with the weaker assumption that some subject matter – thought of as an

equivalence relation on worlds, which we might code as a proper class of ordered pairs of worlds – divides logical space into infinitely many cells and is such that, for every collection of cells of any coarsening of that subject matter, there is a proposition with that coarsened subject matter that is true throughout each of those cells and false throughout all other cells.)

Definition 68 Let \mathcal{A}^Ω be the result of adding NON-TRIVIALITY to the set of all instances of COMMUTATIVITY, DISTRIBUTION, IDENTITY, CONTRADICTION and EQUIVALENCE and closing the result under classical logic.

Claim 69 If propositions form an agglomerative algebra in the manner of the theory of aboutness sketched in Section 7.2, then $\mathcal{L}_{MU} = \mathcal{A}^\Omega$.

Argument: This is not a mathematically precise claim because the sketched theory of aboutness to which it alludes has not been made formally precise. However, it is a consequence of some natural ways of modeling such a theory.

First, consider the metalinguistic structures of Example 61, in the case where C is an infinite set of individual constants (which corresponds to the plausible metaphysical assumption that there are infinitely many individuals). Then \mathcal{A}^Ω is the logic of $\mathbf{A} = \langle A/\sim, \wedge\sim, \neg\sim, \top\sim \rangle$. For suppose $\varphi \notin \mathcal{A}^\Omega$. Then there is a countable model \mathcal{M} of \mathcal{A}^Ω in which φ is false. So there is a finite model \mathcal{M}^- of \mathcal{A}^Ω in which φ is false, consisting of the smallest substructure of \mathcal{M} that forms a non-trivial agglomerative algebra, that interprets all constant occurring in φ the same way that \mathcal{M} does, and that interprets all other constants of \mathcal{L} the same way that it interprets \top . This finite agglomerative algebra can be embedded into \mathbf{A} , because it can be embedded into the subalgebra \mathbf{A}^- of \mathbf{A} based on $A^- = \{x \in S/\sim : [x]_L \cap [\top\sim]_N \neq \emptyset\}$. To see this, note that \mathbf{A}^- is isomorphic to the direct product of \mathbf{A}_L^- and \mathbf{A}_N . Since \mathcal{M}^- is finite, our representation theorems imply that it is isomorphic a subalgebra of a direct product of a finite non-trivial Boolean algebra (which can be embedded into \mathbf{A}_L^- , since any finite non-trivial Boolean algebra can be embedded into any infinite Boolean algebra) and a finite bounded meet-semilattice (which can be embedded into \mathbf{A}_N , since every finite semilattice can be embedded into the semilattice of finite subsets of an infinite set under the operation of set union); it follows that \mathcal{M}^- , considered as an agglomerative algebra, can be embedded into \mathbf{A}^- , and hence into \mathbf{A} .

Second, consider the permutation-based structures of Example 63. Here we can argue that any countable model of \mathcal{A}^Ω can be embedded into any such structure in which V and $\{\langle X, \emptyset \rangle \in A : X \subseteq W\}$ are both infinite, a condition which corresponds to the plausible metaphysical claims that there are infinitely many individuals and infinitely many propositions about no particular individuals. As above, consider the subalgebra of propositions that are logically equivalent to some element non-logically equivalent to \top . This subalgebra is isomorphic the direct product of the Boolean algebra of propositions non-logically equivalent to \top (which is infinite by assumption and complete by construction) and the bounded meet-semilattice of propositions logically equivalent to \top (which by construction is isomorphic to powerset of the set of individuals under the operation of set union). Any non-trivial countable agglomerative algebra can be embedded into such an algebra, because

every such algebra is isomorphic to a subalgebra of a direct product of a non-trivial countable Boolean algebra (which can be embedded into any infinite complete Boolean algebra) and a countable bounded meet-semilattice (which can be embedded into the semilattice of subsets of any infinite set).

Acknowledgments I am grateful to Cian Dorr, Peter Fritz, Harvey Lederman, and Steve Yablo for their extremely helpful comments on earlier versions of this paper, to Branden Fitelson for introducing me to Prover9, and to Arc Kocurek for helping me draw the figure accompanying Example 61.

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